Trends in Mathematics

Piotr Kielanowski
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Editors

Geometric Methods in Physics

XXX Workshop, Białowieża, Poland, June 26 to July 2, 2011

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Preface

The Workshop on Geometric Methods in Physics – the Białowieża Workshop is an annual conference in the fields of mathematical physics and mathematics, organized by the Department of Mathematical Physics of the University of Białystok, Poland. The XXXth Workshop was held during the period June 26–July 2, 2011.

Białowieża, the traditional conference site, is a tiny village in the eastern part of Poland. It is famous for its bison reserve and remaining ancient European primeval forest. The beautiful surroundings help the participants maintain close contact and enjoy a variety of activities together, including excursions and the late evening "campfire", creating a special atmosphere of collaboration and understanding.

The scientific program of the workshop generally covers such subjects as quantization, integrable systems, coherent states, non-commutative geometry, Poisson and symplectic geometry, infinite-dimensional Lie groups and Lie algebras. In 2011, the conference included three special sessions devoted to the achievements of three mathematical physicists: Felix Alexandrovich Berezin, Bogdan Mielnik, and Stanisław Lech Woronowicz, and their impact on present-day research.

Berezin Memorial Session: Representations, Quantization and Supergeometry. Felix Alexandrovich Berezin (1931–1980) made important contributions to such classical subjects as group representation theory, the spectral theory of operators, quantum mechanics, statistical physics, and constructive quantum field theory. He also created new concepts, such as a general approach to the quantization problem, the formulation of second quantization in terms of functional integrals, and especially what became known as "supermathematics", i.e., the theory of supermanifolds and Lie supergroups. More than 30 years after his death, his ideas are still alive and play an important role in mathematical physics. These points are discussed in the special paper included in this volume: "Felix Alexandrovich Berezin and his work" by Alexander Karabegov, Yuri Neretin and Theodore Voronov.

Special session devoted to Bogdan Mielnik. Bogdan Mielnik, the outstanding Polish physicist, turned 75 in 2011. His main line of research has been in the foundations of quantum mechanics. Here, he has always taken an unorthodox and very general approach, based on original ideas such as the convex structure of the space of quantum states or the algebraic manipulation of quantum states. Mielnik has been professor at the Institute of Theoretical Physics of the Warsaw University, and since 1981 has been professor at the Centro de Investigación y de Estudios Avanzados in Mexico City, his current position. His research directions and achievements are described in the special paper included in this volume: "Bogdan Mielnik: contributions to quantum control" by David J. Fernández C.

Special session devoted to Stanisław Lech Woronowicz. We also celebrated the 70th birthday of Stanisław Lech Woronowicz, the outstanding Polish mathematician and mathematical physicist, one of the discoverers of quantum groups (together with V.G. Drinfeld and M. Jimbo). Unlike the algebraic approach to quantum groups, the approach put forward by Woronowicz is based on ideas of functional analysis and operator algebras. Since this volume does not contain a special contribution about Woronowicz's life and activity, we present some information here.

Woronowicz already demonstrated exceptional abilities as an undergraduate student, and was given the position of *Assistant* at Warsaw University even before graduating (M.S.). He joined the Department of Mathematical Methods in Physics, and at the beginning worked on mathematical aspects of quantum theory, axiomatic quantum field theory and operator algebras. In 1968, he received his Ph.D. after presenting the thesis "Causal spaces". In 1972, he received the habilitation (D.Sc.) on the basis of his paper "Foundations of axiomatic quantum field theory". Beginning in 1979, Woronowicz has been mainly interested in the theory of quantum groups, and is regarded as one of its founders.

In 1979, in a talk at the International Conference on Mathematical Physics in Lausanne, he presented the idea and gave the necessary definitions for replacing the commutative C^*-algebra of functions on a compact topological space by a non-commutative algebra, which forms the dual description of the space corresponding to the non-commutative group. Numerous examples implementing these ideas were contained in papers on quantum deformations of groups and spaces published over the next 15 years by Woronowicz and his co-workers. Later Woronowicz also investigated quantum deformations of non-compact groups, such as the group $E(2)$ of motions of Euclidean space, and the Lorentz group. Woronowicz has received many awards, both Polish and international: the Stefan Banach Prize of the Polish Mathematical Society (1972), the Alfred Jurzykowski Prize (New York, 1989), the Prize of the Foundation for Polish Science (1993), and the Humboldt Research Award (2008). Since 1992, he has been a member of the Polish Academy of Sciences. Since 2011, Woronowicz has been professor at the Institute of Mathematics of the University of Białystok.

Acknowledgment. The organizers of WGMP XXX gratefully acknowledge financial support from the University of Białystok, and the European Science Foundation (ESF) Research Networking Programme "Harmonic and Complex Analysis and its Applications" (HCAA). The U.S. National Science Foundation (NSF grant no. 1124929) supported the U.S. participants (which, in particular, allowed a number of young American researchers to attend the meeting). The Russian Foundation for Basic Research (RFBR) supported the participation of mathematicians and physicists from Russia. We would like to thank them all. Last but not the least, the organizers would like to acknowledge the extraordinary amount of work done by students and young researchers from Białystok during the meeting, to make the conference a success.

March 2012 The Editors

Address of Professor Krzysztof Maurin

In 1982, the first Workshop on Geometric Methods in Physics was inaugurated by Professor Krzysztof Maurin, who is the founder of the Department of Mathematical Methods in Physics at Warsaw University. Professor Maurin has been the teacher of many generations of mathematical physicists; his students include Anatol Odzijewicz, the founder of the Białowieża Workshop, and S.L. Woronowicz, the outstanding Polish mathematical physicist.

We invited Professor Maurin to give the opening address at WGMP XXX, but regretfully he was unable to travel due to his fragile health, and consequently could not participate. Nevertheless he sent a special address to the participants, which we include here (translated from the Polish).

Ladies and Gentlemen,

Today we begin the XXXth jubilee conference in Białowieża. Thirty years ago, when I opened the first conference organized by Dr. Anatol Odzijewicz, I could not have known that I was witness to the creation of a very vital structure, a conference series that would become an ongoing meeting point for theoretical physicists and mathematicians.

One other, comparable Polish initiative of this type is the "Copernicus Name Day", which was initiated by Roman Ingarden and his disciples in Toruń. The most famous European forum for mathematicians and physicists may be the conferences in Oberwolfach in Schwarzwald, where for the whole year there takes place a meeting every week devoted to a different subject of mathematics or mathematical physics. Anyone who has attended such international gatherings will never forget them. The Institute in Oberwolfach has, of course, a wonderful library. Białowieża is grateful to Anatol for the extraordinary "skansen" whose creation he has led – proof of his deep devotion to the beautiful landscape, the ancient forest, and the local culture. And at none of the other conferences are there unforgettable night campfires, or soccer games between the participants.

The present XXXth Workshop also has a special character. Three days of the workshop will be devoted to discussion of the achievements of three mathematical physicists: Felix A. Berezin, Bogdan Mielnik and Stanisław Lech Woronowicz. I hope the program will not be overloaded, and that there will also be time for personal contacts. The large number of participants is proof of how popular and highly valued the Workshop is.

With these words I complete my short address, and wish everyone a fruitful and enjoyable conference.

Geometric Methods in Physics. XXX Workshop 2011
Trends in Mathematics, xiii–xix

The Białowieża Workshop on Geometric Methods in Physics: An Impression of Three Extraordinary Decades

Gerald A. Goldin

> *"I cannot see what flowers are at my feet,*
> *Nor what soft incense hangs upon the boughs,*
> *But, in embalmèd darkness, guess each sweet*
> *Wherewith the seasonable month endows*
> *The grass, the thicket, and the fruit-tree wild;*
> *. . . "*
> *John Keats* (1795–1821)*, Ode to a Nightingale*

Abstract. The beauty of nature and an extraordinary spirit of shared scientific inquiry have combined in the environs of Białowieża Forest, leading to an extraordinary workshop series that marks its thirtieth anniversary with this volume.

Mathematics Subject Classification (2010). Primary 01-06; Secondary 51-03.

Keywords. Beauty, geometric methods, physics, primeval forest, workshop.

Each year at the end of June or the beginning of July, the Workshop on Geometric Methods in Physics (WGMP) takes place in Białowieża National Park, the location of the last true lowland primeval forest in Europe. Here ancient trees tower majestically over meadows, lush wetlands, and woodland paths. And here for one week every summer, mathematicians and physicists from all over the world gather to present our work, share ideas, and come to know each other in ways that transcend the ordinary.

In accepting the invitation to write this article, I have been drawn to reflect on my twenty years of participation in the WGMP, and the thirty years altogether during which it has been held.

Partial support was provided by the U.S. National Science Foundation (NSF), grant no. 1124929. Any opinions or conclusions expressed are solely those of the author, and do not necessarily reflect the views of the NSF.

Poland, Europe, and indeed the world have changed in these decades, in the wake of vast political upheavals. In 1995 (at WGMP-XIV), international visitors found that the new złoty had suddenly replaced 10,000 old złotych. Nine years later in 2004 (not long before WGMP-XXIII), Poland became a member of the European Union. By then the border with Belarus, which actually runs through the forest preserve, had effectively become the new political boundary between East and West.

During the 1990s, Warsaw too changed visibly and quickly, and subsequently continued to do so. But change arrived more gradually in the picturesque town of Białowieża. Most of the modest houses have been refurbished over the past 10–15 years. Several small hotels have opened near the park, accommodating the workshop participants in a variety of settings. Yet Białowieża Forest remains serene and timeless, inspiring as ever the creative spirit of those attending the WGMP.

Indeed, in the early morning hours – when the lush landscape is enveloped in mist and (it is said) the *żubry* (European bison) are most likely to allow themselves to be seen – one easily imagines that one has been transported back in time several hundred years. The occasional horse-drawn buggy along a country road contributes to the vividness of this impression.

According to Anatol Odzijewicz at the University of Białystok, a founder and unquestionably the prime mover of the WGMP series, the workshop had a rather local character for most of its first decade (from 1982 to 1990). During this period, the major organizational effort was provided by Odzijewicz and his colleagues at the Institute of Physics in Białystok – Andrzej Kryszen, and Klara Gilewicz (now Janglajew). One of the people mainly responsible for the scientific program at this time was Krzysztof Maurin in the Chair of Mathematical Methods in Physics in Warsaw. Coworkers from both groups (including Stanisław Woronowicz from Warsaw) participated in the workshops.

The shift from a local workshop toward one with significant international participation took place in 1991 (WGMP-X). At that point, members of the organizing committee included Jean-Pierre Antoine, Thomas Friedrich, Jean-Pierre Gazeau, Ivailo Mladenov, and Mikhail Shubin. In particular Mladenov, Gazeau, and others played major roles in inviting non-Polish participants.

As Gazeau recalls, he met Odzijewicz for the first time during a winter school in Srni, former Czechoslovakia, in January 1989. He remembers one evening when Odzijewicz gave a highly-appreciated Belorussian song performance with his guitar. They were both working then on coherent states for the Poincaré group, so they exchanged invitations to visit and Odzijewicz went to Paris for one month in September 1990. When Gazeau came to Białystok in March 1991 he experienced much singing and dancing, and a group picnic in Białowieża in one of the old houses (now part of the open-air museum, or skansen). Back in Paris, Gazeau recruited others, including my close friend S. Twareque Ali, who came to the WGMP for the first time in 1991. For some of the succeeding workshops, Gazeau was even able to secure financial support from the French Embassy in Warsaw for inviting foreign scientists.

Soon colleagues such as Antoine, Stephan De Bièvre, Ugo Moschella, and others became enthusiastic participants, and Ali began to serve as another leading international proponent of the workshop series. The subsequent scientific secretaries of the conference have been Wojciech Lisiecki, Aleksander Strasburger, Piotr Kielanowski, and Tomasz Goliński (who is the current scientific secretary). The organizing committee today (as WGMP-XXXI is being planned) consists of Odzijewicz (chairman), Goliński, Ali, and Kielanowski, as well as Victor Buchstaber, Alina Dobrogowska, Martin Schlichenmaier, Aneta Śliżewska, and Theodore Voronov.

Thus I came to the workshop for the first time in 1992 (WGMP-XI), invited with great enthusiasm by Ali who had attended the year before. The scientific ambience was unlike any other I had experienced – and the WGMP became, for me, a kind of annual, peaceful "fixed point" in the swirl of scientific meetings and academic responsibilities. I might attend other conference series occasionally or frequently, but the week in Białowieża was never to be missed. So I participated for twenty consecutive years.

How can I describe the essence of this ambience? Perhaps the remoteness of the location (four hours' travel from Warsaw), the hospitality of the local organizers, and the silent majesty of the forest combine to generate an unusual openness to intimate conversation and sharing among the participants. Distinctions of academic status, which in many contexts impose social boundaries, rules, restrictions and priorities on our patterns of inquiry – and what Keats called "the weariness, the fever, and the fret" – seem to disappear into the canopy of leaves overhead. And for a full, glorious week we think, talk, envision new possibilities, and (hopefully) discover the best scientific insights dwelling inside us.

Certain themes associated with geometric methods in physics thread like strands of silver through the tapestry of the workshops. As I write, I have before me assorted *Proceedings* from meetings I attended. There are books published through PWN (Polish Scientific Publishers), Plenum, World Scientific, and AIP (American Institute of Physics Conference Series). The contributions of some years are featured in a supplement to *Journal of Nonlinear Mathematical Physics* (WGMP-XXI and WGMP-XXII), and in special issues of the *Journal of Geometry and Symmetry in Physics* (WGMP-XXIV).

Opening a few of these books randomly, one finds that in 1992 (WGMP-XI) there were contributions on geometric quantization, loop spaces and path integral quantization, infinite-dimensional systems and the theory of vortices, Berry phase, and related topics. In 1998 (WGMP-XVII),the highlighted topics include coherent states, wavelets, deformation and geometric quantization, gravity and quantum gravity, and geometrical methods for field theory.

A special volume was published following WGMP-XX, entitled *Twenty Years of Białowieża: A Mathematical Anthology*, edited by Ali et al. [1], containing invited articles on some of the most featured topics of WGMP across its first two decades: diffeomorphism groups and Lie algebras of vector fields, quantization and coherent states, symplectic and Poisson geometry, quantum groups, and other topics. In 2007 (WGMP-XXVI) and 2008 (WGMP-XXVII), major themes include quantization, field theory, Poisson geometry and Hamiltonian systems, noncommutative geometry, integrable systems, Lie algebras, and quantum deformations of groups. Through WGMP-XXX (the present volume) and WGMP-XXXI (announced for June 2012), most of these topics continue to be explored. The permanent website of the conference at http://wgmp.uwb.edu.pl/ contains not only the announcement of the next workshop, but links to posters and photographs from earlier workshops going back nearly twenty years, displaying these and related themes.

Some prominent individual participants are associated with memorable workshop talks and discussions. Among the distinguished mathematicians and physicists with whom we have been privileged to share time in Białowieża, in addition to those already mentioned, have been Dmitri Anosov, Francesco Calogero, Alberto Cattaneo, Bryce DeWitt and Cécile DeWitt-Morette, David Elworthy, Gerard Emch, Boris Fedosov, Moshe Flato, Roy Glauber, John Klauder, Martin Kruskal, Kirill MacKenzie, Varghese Mathai, George Mackey, Bogdan Mielnik, and Alexander Veselov. This abbreviated list omits many more, and only begins to convey the high level of the science. Yet each day, the casual, informal ambience encourages everyone to talk with (and sing with, dance with, and share with) everyone else – from graduate students just starting out to senior scientists with interesting stories to tell.

In fact, the WGMP has consistently subsidized the participation of graduate students. For many graduate students in mathematics and physics, the meeting in Białowieża actually provided the first opportunity to interact seriously with the international mathematics and physics communities.

And it does not stop in the summer. The WGMP international advisory committee draws most of its participants from the invited conference speakers, leading in turn to the creation of informal networks collaborating on various scientific research and education development activities. This is, perhaps, the less visible part of Białowieża's world-wide influence.

For example, Rutgers University formed a partnership with Université d'Abomey-Calavi (Cotonou, Benin), initiated with fellow-advisory board member M. Norbert Hounkonnou, that has led to reciprocal faculty visits including talks and workshops on mathematics education in Benin and the USA, and contributed lectures in Cotonou for graduate students across sub-Saharan Africa. Other examples include numerous research collaborations and student exchanges between Africa, the former Soviet republics, Canada, and European countries such as France, Belgium, and Poland.

A few personal recollections from particular workshops probably typify the experiences of many of the WGMP participants.

In the summer of 1996 (WGMP-XV), Bryce and Cécile DeWitt and George and Alice Mackey came for the week. I had first met George Mackey decades earlier, at the Battelle Seattle 1969 Rencontres, after having been greatly influenced in my undergraduate study and graduate work on current algebras by Mackey's work on induced representations and systems of imprimitivity. Subsequently Cécile DeWitt, in a 1971 paper with Michael Laidlaw, had laid groundwork for my independent development of intermediate statistics for quantum particles in two-space (with Ralph Menikoff and David Sharp at Los Alamos National Laboratory). And Bryce DeWitt had, of course, influenced my entire generation of physics students in our thinking about gravity. So this was a rare opportunity. There occurred an incomparable week of conversation about ways of thinking in mathematics and physics, that affects me to this day.

In a subsequent summer, not too long afterward, my daughter Rebecca (then pursuing graduate study in symplectic geometry at Massachusetts Institute of Technology) attended the WGMP – and found considerable inspiration in learning from Cécile DeWitt about her early experiences as a woman in the field of mathematics.

A number of my scientific collaborations originated at WGMP – for example, with Robert Owczarek and with Shahn Majid. In 2006 (WGMP-XXV), my Rutgers colleague Martin Kruskal came to the Białowieża meeting. Surprisingly, perhaps, this was our first-ever opportunity to talk in depth. Sadly, it turned out to have been our only opportunity, as Kruskal passed away the following winter. His talk at WGMP was not about solitons, but about "surreal numbers" – hardly "geometric methods in physics" – yet it was greatly appreciated by the participants. I recall the long walk we took together on the beautiful path circling the park, discussing surreal numbers and some topics pertaining to the foundations of mathematics.

Then, of course, I remember some fascinating non-mathematical moments that reflect the welcoming informality and rare spirit of the workshops. One summer Nicolaas (Klaas) Landsman gave a spontaneous talk on what an effective scientific presentation should look like, characterizing the best technique as resembling the act of "peeling an onion." The same year, Katherine Brading (then an Oxford philosophy of science graduate student) offered an unplanned, historical/philosophical talk about Emmy Noether, Hermann Weyl, local symmetries and conserved quantities.

Another summer Roger Picken, who had given a talk about braids, knots, and tangles, provided everyone with outdoor lessons in Scottish country dancing to the delight of the group.

And on a different occasion, Carl Bender and I took a walk around the park. Bender, whom I have known since high school when we competed in the same chess league, had presented an interesting survey about the convergence and asymptotic behavior of perturbation series. But what I remember vividly is his reciting to me entirely from memory the long "nonsense" poem by Edward Lear, "The Courtship of the Yonghy-Bonghy Bo," which he recalled from childhood – a surprisingly moving poem to hear, as we walked in the beautiful setting of Białowieża Park.

In short, if the scientific themes are the silver strands in the complex fabric of the Białowieża workshops, then the individual participants – whether returning frequently, or occasionally, or joining the workshop only once – are indeed the golden ones.

I have mentioned a unique attraction, located just at the edge of the village of Białowieża – the skansen, a kind of spacious outdoor museum that includes fields and wetlands, old houses, windmills, and farm and household implements characteristic of an earlier age. Some strikingly beautiful photographs of this re-creation can be found at a various websites, for example,

http://pl.wikipedia.org/wiki/Skansen_w_Białowieży.

As with the WGMP series itself, the existence of this skansen – which has been created over the same thirty-year time period as the workshops – is in large measure due to the energy, patience, and perseverance of Odzijewicz.

Certain activities have become ritual events during the workshops. Early in the week, there is always a bonfire – with music, dancing, Polish sausages and grilled meats, pickles, rye bread, beer and special vodkas as well as softer drinks, and plenty of the delicious cabbage stew called bigos. It takes place in the skansen, where a special structure offers picnic tables, benches, and a little protection in case of rain. Some of the younger workshop participants, or those of us young at heart, stay up nearly till dawn – posing a challenge to attendance at the next morning's talks.

One afternoon later in the week is free, devoted to a traditional guided excursion into the protected part of the forest (or to other leisure activity). Another evening, the workshop banquet takes place – a veritable feast of delicious Polish dishes, in extraordinary variety, again with music and dancing. This time early lectures are not scheduled the next morning, allowing the "night owls" an opportunity to sleep late. This banquet is also the occasion for impromptu recitations of original poetry or limericks, for folk-singing and dancing, and for the offering of toasts – to the organizers, the speakers, the students, and best of all, to Anatol Odzijewicz, whose spirit of scientific inquiry and warmth of friendship have (we have come to understand) infused the atmosphere of the workshop.

One may read more about the WGMP series in the marvelous feature by Ali and Voronov [2] in the European Mathematical Society Newsletter, March 2010, also offering interesting and historic photographs by Goliński; see

http://www.ems-ph.org/journals/newsletter/pdf/2010-03-75.pdf

Finally, at the end of the week, the workshop comes to a close. Saturday night in Białowieża is typically Kupala Night (Noc Kupały), when a festival of regional folk dances and music attracts hundreds of local visitors. And early Sunday morning, as the sun comes up, the special bus stands ready to carry the participants to Warsaw Central Station and to Warsaw Chopin Airport. Our suitcases are packed. Our goodbyes are said with unusual emotion. And our thoughts dwell on the science, the mathematics, the people we have met for the first time, the colleagues with whom we have reconnected, and the awe we have felt walking in the shadows of the ancient trees which grew up long before we were born, and which will endure long after we have gone.

And the final verses of Keats' *Ode to a Nightingale* echo in our minds as we leave the forest,

"Was it a vision, or a waking dream?
Fled is that music: – do I wake or sleep?"

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[1] S.T. Ali, G.G. Emch, A. Odzijewicz, M. Schlichenmaier, and S.L. Woronowicz (eds.), *Twenty Years of Białowieża: A Mathematical Anthology*, World Scientific (2005).

[2] S.T. Ali and T. Voronov, *The Białowieża Meetings on Geometric Methods in Physics: Thirty Years of Success and Inspiration*, in *Newsletter of the European Mathematical Society*, Issue No. 75, March 2010, pp. 10–13.

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Felix Alexandrovich Berezin and His Work

Alexander Karabegov, Yuri Neretin and Theodore Voronov

To the memory of F.A. Berezin (1931–1980)

Abstract. This is a survey of Berezin's work focused on three topics: representation theory, general concept of quantization, and supermathematics.

Mathematics Subject Classification (2010). Primary 01A70; Secondary 58A50, 58Z99, 22E66, 81S99.

Keywords. Representation theory, Laplace operators, second quantization, deformation quantization, symbols, supermanifolds, Lie supergroups.

1. Preface

This text has resulted from our participation in the XXXth Workshop on Geometric Methods in Physics held in Białowieża in summer 2011. Part of this conference was a special Berezin Memorial Session: Representations, Quantization and Supergeometry. F.A. Berezin, who died untimely in 1980 in a water accident during a trip to Kolyma, would have been eighty in 2011.

This is an attempt to give a survey of Berezin's remarkable work and its influence for today. Obviously, we could not cover everything. This survey concentrates on three topics: representation theory, quantization and supermathematics. Outside of its scope remained, in particular, some physical works in which Berezin was applying his approach to second quantization and his theory of quantization. Also, we did not consider two important but somewhat stand-alone topics of the latest period of Berezin's work devoted to an interpretation of equations such as KdV from the viewpoint of infinite-dimensional groups [49, 50] (joint with A.M. Perelomov) and a method of computing characteristic classes [53] (joint with V.S. Retakh).

For a sketch of Berezin's life and personality, we refer to a brilliant text by R.A. Minlos [93].

Sections 2 and 3 below were written by Yu.A. Neretin. Section 4 was written by A.V. Karabegov. Section 5 was written by Th.Th. Voronov, who also proposed the general plan of the paper and made the final editing.

2. Laplace operators on semisimple Lie groups

The main scientific activity of F.A. Berezin was related with mathematical physics, quantization, infinite-dimensional analysis and infinite-dimensional groups, and supermathematics. But in 1950s he started in classical representation theory (which at that time was new and not yet classical).

2.1. Berezin's Ph.D. thesis: characters of complex semisimple Lie groups and classification of irreducible representations

Our first topic[1] is the cluster of papers 1956–57: announcements [7], [8], [40], [9], the main text [10], and an addition in [16]. This work has a substantial overlap with Harish-Chandra's papers of the same years, see [76]. F.A. Berezin in 1956 claimed that he classified all irreducible representations of complex semisimple Lie groups in Banach spaces. We shall say a few words about this result and the approach, which is interesting no less than the classification.

The technology for construction representations of semisimple groups (parabolic induction and principal series) was proposed by I.M. Gelfand and M.A. Naimark in book [70]. On the other hand, Harish-Chandra [75] in 1953 proved the 'subquotient theorem': each irreducible representation is a subquotient of a representation of the principal (generally, non-unitary) series.

Consider a complex semisimple (or reductive) Lie group G, its maximal compact subgroup K and the symmetric space G/K. For instance, consider $G = \mathrm{GL}(n, \mathbb{C})$; then $K = \mathrm{U}(n)$ and G/K is the space of positive definite matrices of order n. A *Laplace operator* is a G-invariant partial differential operator on G/K. Let us restrict a Laplace operator to the space of K-invariant functions (for instance, in the example above it is the space of functions depending on eigenvalues of matrices). The *radial part of Laplace operator* is such a restriction.

Berezin described explicitly the radial parts of the Laplace operators on G/K. He showed that in appropriate coordinates[2] $t_1, \ldots, t_n$ on $K \setminus G/K$ each radial part has the form

$$p\Big(\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_n}\Big), \tag{1}$$

where p is a symmetric (with respect to the Weyl group) polynomial.

The first application was a proof of the formula for spherical functions on complex semisimple Lie groups from Gelfand and Naimark's book [70]. One of the possible definitions of *spherical functions*: they are K-invariant functions on G/K that are joint eigenfunctions for the Laplace operators. I.M. Gelfand and M.A. Naimark proved that for $G = \mathrm{GL}(n, \mathbb{C})$ such functions can be written in the

[1]This was not the first work of Berezin. The paper [39] of Berezin and I.M. Gelfand (1956) on convolution hypergroups was one of the first attacks on the Horn problem; in particular they showed a link between eigenvalue inequalities and tensor products of irreducible representations of semisimple groups, see [86], [69].

[2]We also allow change $f(t) \mapsto \alpha(t) f(t)$.

terms of the eigenvalues e^{t_k} as

$$\Phi_\lambda(t) = \mathrm{const}(\lambda) \cdot \frac{\det_{k,m}\{e^{\lambda_k t_m}\}}{\det_{k,m}\{e^{k t_m}\}} \tag{2}$$

as in the Weyl character formula[3] for finite-dimensional representations of $\mathrm{GL}(n,\mathbb{C})$, but the exponents λ_j are complex. They wrote the same formula for other complex classical groups, but it seems that their published calculation[4] can be applied only for $\mathrm{GL}(n,\mathbb{C})$. Berezin reduced the problem to a search of common eigenvalues of operators (1) and solved it.

Next, consider Laplace operators on a complex semisimple Lie group G, i.e., differential operators invariant with respect to left and right translations on G. We can consider G as a symmetric space, it acts on itself by left and right translations, $g \mapsto h_1^{-1} g h_2$, the stabilizer of the point $1 \in G$ is the diagonal $\mathrm{diag}(G) \subset G \times G$, i.e., we get the homogeneous space $G \times G/\mathrm{diag}(G)$. Note also that $G \times G/\mathrm{diag}(G)$ is the complexification of the space G/K. We again can consider the radial parts of Laplace operators as the restrictions of Laplace operators to the space of functions depending on eigenvalues λ_j. Since now eigenvalues are complex, the formula transforms to

$$p\left(\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_n}; \frac{\partial}{\partial \overline{t}_1}, \dots \frac{\partial}{\partial \overline{t}_n}\right), \tag{3}$$

where p is separately symmetric with respect to holomorphic and anti-holomorphic partial derivatives[5].

Recall that for infinite-dimensional representations ρ the usual definition of the character $\chi(g) = \mathrm{tr}\,\rho(g)$ makes no sense, because an invertible operator has no trace. However, for *irreducible* representations of semisimple Lie groups and smooth functions f with compact supports the operators $\rho(f) = \int f(g)\rho(g)$ are of trace class. Therefore $f \mapsto \mathrm{tr}\,\rho(f)$ is a distribution on the group in the sense of L. Schwartz. This is the definition of the *character* of an irreducible representation.

A character is invariant with respect to the conjugations $g \mapsto hgh^{-1}$. Also, it is easy to show that a character is an eigenfunction of all Laplace operators. The radial parts of Laplace operators were evaluated, so we can look for characters as joint eigenfunctions of operators (3). Algebraically the problem is similar to calculation of spherical functions and final formulas are also similar (but there are various additional analytic difficulties).

For a generic eigenvalue, a symmetric solution is unique. It has the form

$$\sum_{\sigma \in S_n} (-1)^\sigma e^{\sum_k (p_k t_{\sigma(k)} + q_k \overline{t}_{\sigma(k)})},$$

for $G = \mathrm{GL}(n,\mathbb{C})$, here S_n is the symmetric group. This is the character of a representation of the principal series. For 'degenerate' cases there are finite subspaces of

[3]The function α from a previous footnote is the denominator of (2).

[4]It is very interesting, an integration in the Jacobi elliptic coordinates.

[5]The eigenfunctions of (3) are exponential and we have to symmetrize them because we need symmetric solutions.

solutions. Berezin showed that all characters are linear combinations of the characters of representations of principal series. In the introduction to [10], he announced without proof a classification of all irreducible representations. The restriction of a representation of the principal series to K contains a unique subrepresentation with the minimal possible highest weight[6]. We must choose a unique subquotient containing this representation of K.

A formal proof of the classification of representations was not presented in [10], but the theorem about characters and the classification theorem are equivalent[7].

Paper [10] was written in an enthusiastic style and was not always careful. J.M.G. Fell, Harish-Chandra, A.A. Kirillov, and G.M. Mackey formulated two critical arguments; Berezin responded in a separate paper [16].

Firstly, the original Berezin work contains a non-obvious and unproved lemma (on the correspondence between solutions of the systems of PDE in distributions on the group and the system of PDE in radial coordinates). A proof was a subject of the additional paper [16].

Secondly, Berezin actually worked with irreducible representations whose K-spectra have finite multiplicities (i.e., the irreducible Harish-Chandra modules). He formulated the final result as the "classification of all *irreducible* representations in Banach spaces" and at this point he claimed that the equivalence of the two concepts had been proved by Harish-Chandra. But this is not correct[8]. He had to formulate the statement as the "classification of all *completely irreducible*[9] representations in Banach spaces", with the necessary implication proved by R. Godement [71] in 1952.

Recall that the stronger version of classification theorem was proved by Zhelobenko near 1970. For real semisimple groups, the classification was announced by R. Langlands in 1973 and proofs were published by A. Borel and N. Wallach in 1980.

2.2. Radial parts of Laplace operators

Spherical functions, the spherical transform, and the radial parts of Laplace operators appeared in representation theory in the 1950s. Later they became important in integrable systems. On the other hand, they gave a new start for the theory of multivariable special functions (I.G. Macdonald, H. Heckman, E. Opdam, T. Koornwinder, I. Cherednik, and others).

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[6] In 1966 D.P. Zhelobenko and M.A. Naimark [127] announced the classification theorem in a stronger form. Later (1967–1973) D.P. Zhelobenko published a series of papers on complex semisimple Lie groups, e.g., [126], where he, in particular, presented a proof of this theorem (with a contribution of M. Duflo).

[7] It is not difficult to show that the distinct subquotients have different characters. The transition matrix between the characters of the principal series and the characters of irreducible representations is triangular with units on the diagonal.

[8] These two properties are not equivalent, see Soergel's counterexample [117].

[9] There are many versions of irreducibility for infinite-dimensional non-unitary representations. A representation is *completely irreducible* if the image of the group algebra is weakly dense in the algebra of all operators.

Consider a real semisimple Lie group G, its maximal compact subgroup K and the Riemannian symmetric space G/K. If the group G is complex, then the spherical functions are elementary functions, as we have seen above.

But for the simplest of the real groups, $G = \mathrm{SL}(2, \mathbb{R})$, the spherical functions are the Legendre functions. In this case, the radial part of the Laplace operator is a hypergeometric differential operator (with some special values of the parameters). General spherical functions are higher analogs of the Gauss hypergeometric functions. Respectively, the radial parts of the Laplace operators are higher analogs of hypergeometric operators (see expressions in [112] and [77], Chapter 1). The first attack in this direction was made by F.A. Berezin and F.I. Karpelevich [44] in 1958.

Berezin and Karpelevich found a semi-elementary case, the pseudounitary group $G = \mathrm{U}(p, q)$. In this case the radial parts of Laplace operators are also symmetric expressions of the form

$$r\big(L(x_1), \dots, L(x_p)\big)\,,$$

but $L(x)$ is now a second-order (hypergeometric) differential operator,

$$D := x(x+1)\frac{d^2}{dx^2} + \big[(q-p+1) + (q-p)x\big]\frac{d}{dx} + \frac{1}{4}(q-p+1)^2\,.$$

They also evaluated the spherical functions on $\mathrm{U}(p, q)$ as eigenfunctions of the radial Laplace operators. In appropriate coordinates the functions have the form

$$\Phi_s(x) = \mathrm{const}\cdot \frac{\det\limits_{k,j}\Big\{ {}_2F_1\Big[\begin{matrix} \frac{1}{2}(q-p+1)+is_j, \frac{1}{2}(q-p+1)-is_j \\ q-p+1 \end{matrix}; -x_k\Big]\Big\}}{\prod_{1\le k<l\le p}(s_k^2 - s_l^2)\prod_{1\le k<l\le p}(x_k - x_l)}\,.$$

Here ${}_2F_1[\dots]$ is the Gauss hypergeometric function, $x_1, \dots, x_p$ are coordinates on the Cartan subgroup of $\mathrm{U}(p, q)$, and $s_1, \dots, s_p$ are parameters of spherical functions.

This paper was accepted by *Doklady* in June 1957. Near that time Berezin's scientific interests had changed and he left the classical representation theory[10,11].

(The next step was done by M.A. Olshanetsky and A.M. Perelomov [101] in 1976; see also [102]. They wrote the radial part of the second-order Laplace operator. Quite soon J. Sekiguchi [112] obtained a general formula for the groups GL.)

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[10] In 1976 paper [27] and the five ITEP preprints of 1977 included in the English version of [38], Berezin returned to the study of Laplace operators and considered the radial parts of Laplace operators for Lie supergroups (see Subsection 5.5). They are usual (non-super) partial differential operators. This topic is not well understood up to now; A.N. Sergeev and A.P. Veselov produced from this standpoint new operators of Calogero–Moser type whose eigenfunctions are super-Jack functions (which also are functions of even variables), see [113]. On an analog of the group case, see [78].

[11] In 1970s, Berezin made a work on the harmonic analysis in Hilbert spaces of holomorphic functions [23], [26], [34] for a discussion of this work and its continuations, see [118], [97], and [99], Chapter 7.

3. Method of second quantization

Our next topic is the famous book "The method of second quantization" [14] (and the announcements [11, 48, 12, 13]). A more detailed discussion of the intellectual history of this work and its influence is in [98].

3.1. Prehistory

It is known that at the end of 1950s Berezin started to learn physics and to participate in theoretical physics seminars in Moscow. He had to decide between numerous possible ways in this new world and his choice was the problem about the automorphisms of the canonical commutation and anticommutation relations formulated in the book 'Mathematical aspects of the quantum theory of fields' by K.O. Friedrichs [68] of 1953.

Let $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ be self-adjoint operators in a Hilbert space satisfying the conditions

$$[P_k, P_l] = [Q_k, Q_l] = 0, \qquad [P_k, Q_l] = i\delta_{k,l} \tag{4}$$

and without a common invariant subspace. Such conditions are called the *canonical commutation relations*, abbreviation CCR. According to the Stone–von Neumann theorem, such a system of operators is unique up to a unitary equivalence (for a precise forms of the theorem, see, e.g., [15]). In fact, our Hilbert space can be identified with $L^2(\mathbb{R}^n)$ and the operators with x_k, $i\frac{\partial}{\partial x_k}$, respectively. Now let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a symplectic $2n \times 2n$ matrix. Evidently, the operators

$$P'_k = \sum_l \alpha_{kl} P_l + \sum_l \beta_{kl} Q_{kl}, \qquad Q'_k = \sum_l \gamma_{kl} P_l + \sum_l \delta_{kl} Q_{kl} \tag{5}$$

satisfy the same relations (4). Therefore there is a unitary operator $U = U(g)$ such that [12]

$$P'_k = U(g) P_k U(g)^{-1}, \qquad Q'_k = U(g) Q_k U(g)^{-1} \,. \tag{6}$$

By a version of the Schur Lemma, this operator is unique up to a scalar factor. It is easy to see that

$$U(g)U(h) = \lambda(g,h)U(g,h)\,,$$

where $\lambda(\cdot,\cdot)$ is a complex scalar. Apparently, Friedrichs decided that there was nothing to discuss here and asked what would happen if the number n of the operators were ∞. He showed that there are many nonequivalent representations of CCR besides the well-known Fock representation. Next, Friedrichs asked, for which symplectic matrices the system of operators P'_k, Q'_k are equivalent to P_k, Q_k. He formulated a correct conjecture and tried to find explicit formulas for $U(g)$.

[12] Now the mapping $g \mapsto U(g)$ is called the *Weil representation*, see A. Weil's paper [125], 1964. The term is common and convenient, but historically it was a construction due to K.O. Friedrichs and I. Segal.

3.2. Operators and divergences

Consider the usual Fourier transform $\mathcal{F}$ in $L^2(\mathbb{R})$, $\widehat{f}(\xi) = \int e^{ix\xi} f(x)\,dx$. Its definition is not completely straightforward, since the integral can be divergent, and some regularization dance is necessary. If we want to find $\mathcal{F}^2$, we must calculate the kernel

$$K(x,y) = \int e^{iy\xi} e^{ix\xi}\,d\xi$$

Since we know the answer, we can believe that it is obvious. In any case, the integral diverges ...

These difficulties are usual for the work with integral operators in $L^2(\mathbb{R}^n)$. Field theory requires functions of infinite number of variables and passing to the limit $n \to \infty$ only multiplies the problems. Berezin noticed that in the space F_n of entire functions on $\mathbb{C}^n$ with the inner product

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int f(z)\overline{g(z)} e^{-|z|^2}\,d\Re(z)\,d\Im(z)$$

we can realize our operators as

$$P_k = \frac{1}{\sqrt{2}}\Bigl(z_k + \frac{\partial}{\partial z_k}\Bigr), \qquad Q_k = \frac{1}{\sqrt{2}i}\Bigl(z_k - \frac{\partial}{\partial z_k}\Bigr).$$

Therefore this space can be identified with $L^2(\mathbb{R}^n)$. Berezin observed that in the space F_n any bounded operator is an integral operator of the form

$$Af(z) = \int_{\mathbb{C}^n} K(z, \overline{u}) f(u) e^{-|u|^2} du\,d\overline{u}$$

and the integral is convergent. Also the kernel of a product of integral operators is defined by a convergent integral. Next, Berezin showed that this 'holomorphic model' perfectly survives as $n \to \infty$ (only the case $n = \infty$ is discussed in book [14], Berezin uses the term 'generating functional' for the function assigned to an operator). In particular, we can work with bounded operators without any divergences.

Certainly, we need also unbounded operators, where divergent expressions have to appear. But, again, the 'level of divergences' is minimal.

In parallel, Berezin proposed an almost equivalent formalism of Wick symbols. Algebraically, they looks similar to the well-known since 1930s expressions of operators as $A = p(x, \frac{\partial}{\partial x})$, where all x's are at the left and all $\frac{\partial}{\partial x}$'s are at the right. But only few operators can be written in this form if we understand 'functions' literally. In contrast, we can express an operator as $A = p(z, \frac{\partial}{\partial z})$ more or less always.

3.3. Weil representation

Using this operator formalism, Berezin wrote explicit formulas for the operators $U(g)$. He interpreted the conditions (6) as a first-order system of PDEs for the kernels K of $U(\cdot)$, solved the equations and got the expressions of the form

$$K(z, \overline{u}) = \exp\bigl\{S(z, \overline{u})\bigr\}, \tag{7}$$

where S is an explicit quadratic form. Thus we obtain a projective representation of an infinite-dimensional symplectic group by integral operators acting in the space of functions of infinite number of variables. We also can replace $\infty \mapsto n$ and obtain a construction that was completely new in that time.

In particular, Berezin proved the Friedrichs conjecture about the domain of definition of this representation.

3.4. Fermionic Fock space

For us a fermionic Fock space is a space of functions of anticommuting variables. This idea, now common, originated from Berezin's book [14]. Berezin also found that there is a natural integral over anticommuting variables ([11]). We say more about that in Section 5. Berezin showed that an operator in the fermionic Fock space is determined by a function (the 'generating functional') depending on a double collection of anticommuting variables and that it is convenient to express operators in a fermionic Fock space as integral operators, with respect to that peculiar integral.

In [68], Friedrichs also formulated a problem about the *canonical anticommutation relations* (abbreviation CAR)

$$\{P_k, P_l\} = \{Q_k, Q_l\} = 0, \qquad \{P_k, Q_l\} = i\delta_{k,l} \tag{8}$$

and their symmetries (5). Now the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is orthogonal. Berezin solved this problem as well and wrote a formula for the kernels of $U(\cdot)$,

$$K(\xi, \eta) = \exp\{S(\xi, \eta)\}\,, \tag{9}$$

where S is an explicit quadratic expression. Note that the formulas for S in (7) and (9) are similar.

In fact, both theorems are results in the representation theory of infinite-dimensional Lie groups. Berezin's book can be regarded as a mathematization of field theory. However, it was also (Chapters 2 and 3) the first book on infinite-dimensional groups and the start of this theory. For a more detailed discussion, see [98].

3.5. History and references

Main Berezin's results with outlined proofs were announced in *Doklady* paper [11], of March 1961 (accepted in November 1960). The text was written in the telegraphic style usual for *Doklady* of that time: the allowed four pages were all used up to one line. In September 1962, Berezin submitted a large paper to *Uspehi* (that is, *Russian Mathematical Surveys*). The paper was rejected. In the following years, Berezin published more short announcements: [12], [13] and [48]. In 1965[13] the book "The method of second quantization" was published, addressed to physicists[14].

[13]The English version appeared in 1966.

[14]In spite of its physical language, the book is a rigorous, maybe not detailed, mathematical text.

Friedrichs's questions also attracted Irving Segal, who had worked in mathematical field theory since the beginning of 1950s. (In particular, Segal introduced a model of the Fock space as L^2 on a Gaussian measure [110], 1956; later J. Feldman [67], 1959, constructed the action of an infinite-dimensional GL on that space.) In 1959, Segal obtained explicit formulas for the 'Weil representation' for finite n in the space L^2, [111]. In 1961, he proposed a holomorphic model for the boson Fock space (this was also done by V. Bargmann [2] in the same year). In 1962, D. Shale [114] published the solution of the Friedrichs problem for CCR, and in 1965, D. Shale and W.F. Stinespring published their solution for CAR[15] [115].

However these papers did not cover Berezin's results. His book and Berezin himself immediately became famous.

3.6. Berezin's book in physics

Besides the formal results concerning CCR and CAR, the interest of physicists to this text had two additional reasons.

First, the new operator formalism (both bosonic and fermionic) was very convenient. It became easier to write formulas and to calculate.

The second reason was the mysterious parallelism between the bosonic and fermionic spaces which was emphasized in the book. For Berezin himself this was the starting point of his work leading to the creation of supermathematics (see Section 5).

4. Berezin's general concept of quantization

One of the main directions of Berezin's research was mathematical formulation of the concept of quantization as a deformation of a classical mechanical system. In [18] Berezin interpreted the universal enveloping algebra of a Lie algebra as a quantization of the Poisson algebra of polynomial functions on the dual of the Lie algebra. In [24] and [25] Berezin introduced a general concept of quantization based upon algebras of operator symbols depending on a small parameter.

Quoting from [59], "according to the main idea of these works, quantization has the following precise mathematical meaning: the algebra of quantum observables is a deformation of the algebra of classical observables, so that the Planck constant plays the role of the deformation parameter and the direction of deformation (the first derivative in the parameter at zero) is given by the Poisson bracket".

In [26], Berezin studied quantization of complex symmetric spaces. The operator symbols used in quantization were introduced and studied in [20], [21], and [22]. In [34], Berezin obtained the spectral decomposition of the operator connecting covariant and contravariant symbols on classical complex symmetric spaces, now called the Berezin transform. In [19] and [37], Berezin constructed finite approximations of Feynman path integrals with the use of operator symbols. See

[15]Note that the famous mathematicians D. Shale, W.F. Stinespring and J. Feldman were all I. Segal's students.

also Berezin and M.A. Shubin [54] and their joint book "The Schrödinger Equation" [55], which Shubin prepared for publication after Berezin's death.

Let us consider these works in greater detail.

4.1. Poisson bracket and quantization on the dual of a Lie algebra

In the fundamental paper [18] Berezin constructed an integral transform δ from generalized functions on a neighborhood of the identity of a Lie group G to functions on the dual $\tilde{\mathfrak{G}}$ of its Lie algebra $\mathfrak{G}$ and expressed the symmetrization mapping Λ from the symmetric algebra S of $\mathfrak{G}$ to the universal enveloping algebra $\hat{S}$ of $\mathfrak{G}$ through the mapping δ. Let $\{\hat{x}_p\}$ be a basis in $\mathfrak{G}$, $\{t_p\}$ the corresponding coordinates on $\mathfrak{G}$, $\{y_p\}$ the dual coordinates on $\tilde{\mathfrak{G}}$. The mapping δ is defined as follows:

$$\delta : s(g) \mapsto \overset{\delta}{s}(y) = \int e^{-ity} s(g(t))\rho(t)dt, \quad ty = \sum_p t_p y_p,$$

where $t \mapsto g(t)$ is the exponential mapping and $\rho(t)$ is the density of the right-invariant measure on G in the canonical coordinates $\{t_p\}$. The symmetric algebra S of $\mathfrak{G}$ is identified with the space of polynomials on $\tilde{\mathfrak{G}}$ and Λ maps y_p to $\hat{y}_p = -i\hat{x}_p$. The generalized functions supported at the identity of G form an algebra with respect to the convolution. This algebra is naturally identified with the universal enveloping algebra $\hat{S}$. Berezin proved that under this identification the inverse mapping Λ^{-1} is given by the mapping δ. The mapping δ allows to transfer the convolution of generalized functions supported on a small neighborhood U of the identity of the group G to an operation on functions on the dual $\tilde{\mathfrak{G}}$ of the Lie algebra $\mathfrak{G}$. Berezin gave an integral formula for this operation. Given generalized functions $s_1,\ s_2$ supported on U and their convolution s, set $\sigma_1 = \overset{\delta}{s}_1, \sigma_2 = \overset{\delta}{s}_2$ and $\sigma = \overset{\delta}{s}$. Then

$$\sigma(y) = \int K_U(y|y_1, y_2)\sigma_1(y_1)\sigma_2(y_2)dy_1dy_2,$$

where

$$K_U(y|y_1, y_2) = \frac{1}{(2\pi)^{2n}} \int_{g(t_1)\in U,\, g(t_2)\in U} e^{-iy\log(g(t_1)g(t_2))+iy_1t_1+iy_2t_2}dt_1dt_2.$$

Berezin noted that this integral formula can be extended to the space S of polynomials on $\tilde{\mathfrak{G}}$ and the resulting algebra is isomorphic to the universal enveloping algebra $\hat{S}$ of $\mathfrak{G}$. Moreover, the leading term of the commutator of polynomials leads to a natural Poisson bracket on the dual of the Lie algebra $\mathfrak{G}$. For arbitrary smooth functions on $\tilde{\mathfrak{G}}$ it is possible to write

$$\{f_1, f_2\} = \sum C_{ij}^k y_k \frac{\partial f_1}{\partial y_i}\frac{\partial f_2}{\partial y_j},$$

where C_{ij}^k are the structure constants of the Lie algebra $\mathfrak{G}$. About the same time, this Poisson bracket on $\tilde{\mathfrak{G}}$ (in the form of a symplectic structure on the coadjoint orbits) was discovered in the orbit method. Therefore it became known as the *Berezin–Kirillov* or *Berezin–Kirillov–Kostant bracket.* (Later Alan Weinstein found out that the bracket had been known already to S. Lie, so the name 'Lie–Poisson bracket' became more standard.)

Thus the universal enveloping algebra of $\mathfrak{G}$ can be interpreted as a quantization of the corresponding Poisson algebra on $\tilde{\mathfrak{G}}$ consisting of the polynomial functions endowed with the Berezin–Kirillov–Kostant Poisson bracket.

In fact, by rescaling this operation by a formal parameter $\hbar$, one can obtain from above Berezin's formula the following integral formula for what is now known as the 'Baker–Campbell–Hausdorff star product' on $\tilde{\mathfrak{G}}$:

$$(f_1 * f_2)(y) = \frac{1}{(2\pi\hbar)^n} \iint dy_1\, dt_1\, dy_2\, dt_2\; f_1(y_1)\, f_2(y_2)\; e^{-\frac{i}{\hbar}(\langle t_1, y_1\rangle + \langle t_2, y_2\rangle - \langle H(t_1,t_2), y\rangle)}. \tag{10}$$

Here $H(t_1, t_2)$ is the formal BCH power series on $\mathfrak{G}$ and the integration extends over $\tilde{\mathfrak{G}} \times \mathfrak{G} \times \tilde{\mathfrak{G}} \times \mathfrak{G}$. The functions f_1 and f_2 can be arbitrary smooth functions on $\tilde{\mathfrak{G}}$ due to the presence of a formal parameter $\hbar$ [16].

4.2. General concept of quantization as deformation

In [24] and [25] Berezin gave a general definition of quantization of a Poisson manifold $(M, \{\cdot,\cdot\})$ as an algebra $(\mathfrak{A}, *)$ of sections of a field of noncommutative algebras $(\mathcal{A}_h, *_h)$ parameterized by the elements h of a set E of positive numbers that has zero as an accumulation point. The Correspondence Principle for this quantization is expressed in terms of a homomorphism

$$\varphi_0 : \mathfrak{A} \to C^\infty(M) \tag{11}$$

such that for $f, g \in \mathfrak{A}$,

$$\varphi_0 \left(\frac{1}{h}(f * g - g * f) \right) = i\{\varphi_0(f), \varphi_0(g)\}.$$

Then he considered a special case when $\mathcal{A}_h \subset C^\infty(M)$, the elements of $\mathfrak{A}$ are functions $f(h,x)$ on $E \times M$, and

$$\varphi_0(f) = \lim_{h\to 0} f(h,x).$$

4.3. Berezin's quantization using symbols

Berezin studied a number of examples of such special quantizations where $\mathcal{A}_h$ for a fixed h is an algebra of symbols of operators in a Hilbert space. To this end, Berezin introduced *covariant* and *contravariant symbols* related to an overcomplete family of vectors in a reproducing kernel space. Namely, consider a Hilbert space H and a set M with measure $d\alpha$ whose elements parameterize a system of vectors $\{e_\alpha\}$ in H. Let P_α be the orthogonal projection operator onto e_α and

$$d\mu(\alpha) = ||e_\alpha||^2 d\alpha$$

[16] We obtained formula (10) around 1998 (Th.V., unpublished) and then realized that it can be deduced from Berezin [18].

be another measure on M. The vectors $\{e_\alpha\}$ form an overcomplete family in H if

$$\int P_\alpha \, d\mu(\alpha) = E$$

is the identity operator in H. Then H is isometrically embedded into $L^2(M, d\alpha)$ by the mapping $H \ni f \mapsto \langle f, e_\alpha \rangle$. The projectors P_α are used to define covariant and contravariant symbols of operators in H as follows. The covariant symbol of an operator $\hat{A}$ is the function

$$A(\alpha) = \operatorname{tr} \hat{A} P_\alpha$$

on M. A function $\overset{\circ}{\mathrm{A}}(\alpha)$ on M is a contravariant symbol of $\hat{A}$ if

$$\hat{A} = \int P_\alpha \overset{\circ}{\mathrm{A}}(\alpha) d\mu(\alpha).$$

The measure μ defines a trace functional on appropriate classes of covariant and contravariants symbols that agrees with the operator trace (see [21]),

$$\operatorname{tr} \hat{A} = \int A \, d\mu = \int \overset{\circ}{\mathrm{A}} \, d\mu.$$

The covariant and contravariant symbols A and $\overset{\circ}{\mathrm{A}}$ of the same operator $\hat{A}$ are connected via the Berezin transform I,

$$A(\alpha) = (I \overset{\circ}{\mathrm{A}})(\alpha) = \int \operatorname{tr}(P_\alpha P_\beta) \overset{\circ}{\mathrm{A}}(\beta) d\mu(\beta).$$

An overcomplete system of vectors $\{e_\alpha\}$ in H may admit a symmetry group G that acts upon H by a unitary representation $g \mapsto U_g$ and upon M by transformations preserving the equivalence class of the measure $d\alpha$ so that

$$U_g e_\alpha = s(\alpha, g) e_{g\alpha},$$

where $s : M \times G \to \mathbb{C}$ is a measurable cocycle satisfying

$$\frac{dg\alpha}{d\alpha} = |s(\alpha, g)|^2.$$

Then $U_g P_\alpha U_g^{-1} = P_{g\alpha}$, the measure $d\mu$ is G-invariant, the symbol mappings

$$\hat{A} \mapsto \operatorname{tr} \hat{A} P_\alpha \text{ and } \overset{\circ}{\mathrm{A}}(\alpha) \mapsto \int P_\alpha \overset{\circ}{\mathrm{A}}(\alpha) d\mu(\alpha)$$

are G-equivariant and the Berezin transform I is G-invariant.

Berezin studied spectral properties of covariant and contravariant symbols in [21] and then used algebras of covariant symbols to define a quantization of a special class of Kähler manifolds in [24] using the saddle-point method. He started with a Kähler manifold M of complex dimension m with a Kähler form ω and the Liouville measure ω^m. He assumed that there exists a global Kähler potential Φ of the form ω and introduced an h-parameterized family of measures

$$d\alpha_h = \mathrm{e}^{-\frac{1}{h}\Phi} \omega^m$$

on M. Then he considered the Hilbert space H_h of holomorphic functions on M square integrable with respect to the measure $d\alpha_h$. The Bergman reproducing kernel of H_h defines an overcomplete system of vectors $\{e_\alpha^{(h)}\}$ in H_h. In order to prove the Correspondence Principle, Berezin imposed a severe assumption on M that

$$e_\alpha^{(h)}(z) = c(h)\mathrm{e}^{\frac{1}{h}\Phi(z,\bar{w})}$$

for $\alpha = (w, \bar{w}) \in M$ and some constant $c(h)$. This assumption is satisfied on Kähler manifolds with a transitive symmetry group which allowed Berezin to quantize complex symmetric spaces (see [26]).

4.4. Influence of Berezin's work

In the following decades Berezin's work on quantization attracted a lot of attention. His results were expanded and generalized by many mathematicians and mathematical physicists in two major directions. First, Berezin's definition of quantization in the special case when $\mathcal{A}_h \subset C^\infty(M)$ was extended to incorporate deformation quantization of Flato et al. [6] as a formal asymptotic expansion in h of the product $*_h$ in the algebra $\mathcal{A}_h$. In the general case this can be achieved by extending the homomorphism (11) in the general definition of quantization to a homomorphism

$$\varphi = \varphi_0 + \nu\varphi_1 + \cdots : \mathfrak{A} \to C^\infty(M)[[\nu]]$$

to the star-algebra of some formal deformation quantization on the Poisson manifold $(M, \{\cdot,\cdot\})$ such that $\varphi(hf) = \nu\varphi(f)$[17]. Examples of such quantizations of Kähler manifolds were first given in [96] and [63, 64]. The second direction was to remove the restrictions on the Kähler manifold in Berezin's quantization. Based on the microlocal technique developed by Boutet de Monvel and Guillemin in [62], it was shown in [60] that Berezin–Toeplitz quantization[18] on general compact Kähler manifolds satisfies an analog of the Correspondence Principle. Then in [106] the existence of the corresponding Berezin–Toeplitz star product was established. In [80] all star products "with separation of variables" on an arbitrary Kähler manifold were classified and in [81] the Berezin–Toeplitz star product was completely identified in terms of this classification. In [66] M. Engliš showed the existence of Berezin star-product on a quite general class of inhomogeneous complex domains. Berezin–Toeplitz quantization was recently studied by microlocal methods developed in [89] and [65]. Applications of Berezin–Toeplitz quantization in the topological quantum field theory were given in [1].

Much work has been done to generalize Berezin's quantization on Kähler manifolds to other spaces. Berezin's first doctoral student Vladimir Molchanov developed harmonic analysis and quantization on para-Hermitian symmetric spaces (see [94], [95]). Berezin's quantization on quantum Cartan domains was considered in [116]. Berezin's quantization was generalized to supermanifolds in [61], [73]. In

[17] Here ν is a formal parameter.

[18] *Berezin–Toeplitz quantization* is defined in terms of operators with given contravariant symbols. Such operators are generalizations of Toeplitz operators.

the framework of this publication it is impossible to give a comprehensive survey of the growing body of papers building upon Berezin's work on quantization and many important papers are inevitably left out.

5. Supermathematics

5.1. Introductory remarks

Without doubt, Berezin is *the* creator of supermathematics, though it was not him who introduced the name. (More about the origin of the name in 5.4.) In hindsight, it is possible to trace the origins of what became supermathematics in various areas of pure mathematics and theoretical physics, but it only due to Berezin's vision and his conscious effort that these previously disjoint pieces became parts of a great unified picture together with a lot of new mathematics discovered by Berezin himself and by those who followed him. Speaking about Berezin's work in supermathematics, it is worth pointing out that it was interrupted by his untimely death when supermathematics was still in the early stages of its development; therefore, the loss caused by Berezin's sudden departure was greater for supermathematics than for other areas of his work.

Berezin's publications related to supermathematics can be divided into two groups corresponding to the two periods: the gestation period (1961–1975) and the 'super' period[19] (1975–1980).

We can formulate the main idea of supermathematics as follows. The systematic consideration of $\mathbb{Z}_2$-graded objects such as Abelian groups, vector spaces, algebras and modules with the corresponding sign convention ("Koszul's sign rule"[20]) allows to construct a natural extension of the 'usual' linear algebra including generalizations of commutative algebras and Lie algebras. This goes further to the extension of differential and integral calculus of many variables and, geometrically, to the extensions of the notions of differentiable manifold, Lie group, algebraic variety (or scheme) and algebraic group.

Two things should be said.

Firstly, Berezin came to his program of supermathematics (without such a name, which appeared later) motivated by physics, more precisely, by his studies of the formalism of second quantization, which lies in the foundations of quantum field theory. The influence of physics was also decisive for the passage of supermathematics from its gestation stage to the modern stage.

Secondly, the 'supermathematical' generalization of the usual notions is *not arbitrary*, but indeed reflects the nature of things: the 'superanalogs' of various objects fit together in the same way as their prototypes do (but may also show non-

[19] The 'super' period is marked by the emergence of the names such as supermanifold, superalgebra, etc. As the borderline we may take the discovery of supersymmetric physical models followed by the introduction of the notion of a supermanifold in mathematics. This division is partly conventional.

[20] From an abstract viewpoint, the sign rule used in supermathematics is a very special example of a "commutativity constraint" or "braiding" in tensor categories.

trivial new phenomena). Moreover, this generalization is *rigid and unique.* There are no known further generalizations based on other gradings or more complicated commutativity constraints. That is, there are quantum groups and quantum spaces; however, they are isolated examples unified philosophically but not by a general theory such as a (non-existing) 'quantum' or 'braided' geometry, although these terms are sometimes applied.

5.2. Analysis on a Grassmann algebra

As already said, Berezin's program of supermathematics has its roots in his book [14] and the related articles [11, 48, 12, 13]. (See the historical remarks in 3.5.) In order to construct a 'calculus of functionals' for the Fermi fields that can be parallel to the functional calculus used for describing the Bose fields, Berezin introduced differentiation and integration on a Grassmann algebra. He did it first for the Grassmann algebras with finite numbers of generators and then extended the results to the infinite-dimensional 'functional' case that he needed for his problem. This calculus allowed Berezin to obtain a 'functional realization' of the fermionic Fock space similar to the realization of the bosonic Fock space by holomorphic or antiholomorphic functions (or functionals) and to construct the spinor representation of the canonical transformations in the fermionic case (i.e., the spinor representations of certain infinite-dimensional versions of the orthogonal group). These representations were discussed in Section 3 and we shall not repeat it here.

A striking feature of Berezin's calculus on a Grassmann algebra was, as he noted in [14], that *"differently from the usual rule for a change of variables, the independent variables and their differentials transform by reciprocal matrices"*. Here Berezin refers to his formulas

$$\int x\, dx = 1\,, \quad \int dx = 0$$

(where x is a Grassmann generator), which imply $d(ax) = \frac{1}{a}\, dx$. The 'differential dx' is the quantity appearing under the integral sign, which we would now call the Berezin volume element and denote by Dx to distinguish it from the genuine differential of the variable x. We can see here the origins of superdeterminant (now called Berezinian), which was discovered some years later [21]. Another remarkable fact noted and used by Berezin in [14] was the appearance of the (square root of the) determinant of the quadratic form for a 'fermionic Gaussian integral' in the numerator, not in the denominator:

$$\int e^{\sum a_{ik} x_i x_k}\, dx_n \ldots dx_1 = (\det \|2a_{ik}\|)^{1/2}\,,$$

where $a_{ik} = -a_{ki}$, in a sharp contrast with the familiar ("bosonic") case (equation 3.16 in [14]).

Integration on a Grassmann algebra introduced by Berezin was soon applied by Faddeev and Popov in their famous work on quantization of the Yang–Mills

[21] When only odd variables are present, the Berezinian of their linear homogeneous transformation reduces simply to the inverse of the determinant.

field: they expressed the Jacobian factor arising from the separation of the gauge degrees of freedom by a fermionic Gaussian integral over 'ghosts' (the "Faddeev–Popov ghosts") and thus they were able to deduce the Feynman rules including ghosts as following from a local Lagrangian field theory. By contrast, Bryce DeWitt, who obtained close results at the same time, did not know Berezin's integration and because of that failed to arrive to such a natural formulation[22].

At this point it makes sense to discuss the question about Berezin's predecessors. It is sometimes claimed that the use of anticommuting variables for the classical description of fermions was familiar to quantum physicists since 1950s (and hence Berezin did not introduce anything particularly new). Typically Schwinger's name is mentioned in this regard. In reality, the ideas of Schwinger and his disciples such as DeWitt about anticommuting variables were quite vague and did not go any further than the introduction of 'left' and 'right' derivatives with respect to generators of a Grassmann algebra (see, e.g., [108, 109]). Partial differentiation with respect to exterior generators had been already known to Élie Cartan in connection with his method of *repère mobile* (physicists were probably unaware of that). The novelty of Berezin's work in comparison with earlier and simultaneous works by physicists was in the mathematical clarity and power with which he developed the analogy between usual functions and elements of a Grassmann algebra, but the main new feature was integration over Grassmann generators with its striking properties. There is a saying[23] that "derivatives are algebra; analysis begins with integrals". It was Berezin who made this decisive step. It took some time for this achievement to be absorbed by the physical community: the example of DeWitt is a clear evidence.

One person who can be counted as a true predecessor of Berezin, is the British physicist J.L. Martin. In two papers [90, 91] of 1959, Martin introduced the notion of a general Hamiltonian system on a Poisson manifold (in the modern terminology) and suggested to extend it to more general algebras; in particular, he showed how to introduce what we would call a Poisson superalgebra structure on a Grassmann algebra and applied that to obtain a Lagrangian classical counterpart of a quantum particle of spin $1/2$; in the second paper, he started from a general algebraic formalism linking matrix calculus with nilpotent variables and applied it to constructing a Feynman integral over histories for fermionic systems. With hindsight, we may observe that Martin in these two works published together introduced the integral over anticommuting variables. Strangely, he applied the name 'integral' only for the functional case treated in [91]. For the finite-dimensional case, he spoke about cosets *modulo* total differentials in [90] or an unnamed 'operation S_λ' in [91], λ being a Grassmann algebra generator or, more generally, an abstract nilpotent variable. Martin did not consider transformations of variables and the corresponding properties of the integral. It is amazing that the remarkable works [90, 91] were not continued and remained completely unnoticed. (Berezin learned about them only around 1976, see [46]. He gives a very generous reference to [90] in the first sentence of [35].)

[22] DeWitt explicitly admits that in the preface to the Russian translation (1985) of his influential book 'The dynamic theory of groups and fields'.
[23] I learned it from A.A. Kirillov.

5.3. From Grassmann algebras to supermanifolds

Berezin's calculus on a Grassmann algebra as constructed in [11, 48, 12, 13] was not yet supermathematics in the proper sense. Ordinary variables and Grassmann algebra generators were considered in parallel but still separately. There was no mixture of them nor transformations of Grassmann variables other than linear. However, as Berezin described it later, *"the striking coincidence of the main formulas of the operator calculus in the Fermi and Bose variants of the second quantization method ... led to the idea of the possibility of a generalization of all the main notions of analysis so that generators of a Grassmann algebra would be on an equal footing with real or complex variables"* [35, 38].

This was the **program of supermathematics** [24].

The main steps of its implementation were as follows.

In [17], Berezin considered non-linear transformations of anticommuting variables in a clear departure from the standard viewpoint on the exterior algebra as a $\mathbb{Z}$-graded object associated with a linear space. Now the emphasis is shifted to the algebra itself and the transformations are supposed to preserve only $\mathbb{Z}_2$-grading and not necessarily $\mathbb{Z}$-grading. Berezin studied the effect of such transformations on the integral over anticommuting variables and proved that there appears the *inverse* of the determinant of the Jacobi matrix. This was a generalization of the formula in [14] and a step towards the discovery of Berezinian. No mixture with "ordinary" variables yet, but the whole logic leads in this direction.

Algebras generated by even and odd variables appeared in a joint paper of Berezin and G.I. Kac [43], who introduced – in modern language – formal Lie supergroups and Lie superalgebras and established their 1–1 correspondence. They used the results of Milnor and Moore [92]. It should be said that a version of Lie superalgebras where the $\mathbb{Z}_2$-grading arises as the reduction of a $\mathbb{Z}$-grading *modulo* 2 had been long familiar to topologists and differential geometers under the confusing name of "graded Lie algebras" [25]. The understanding that graded (co)commutative Hopf algebras played the role of the corresponding group objects was topologists' folklore [26]. In algebraic topology, Hopf algebras arise as homology or cohomology of topological spaces, so in that context $\mathbb{Z}$-grading is natural. Unlike that, the algebras considered in [43] were supposed to play the role of algebras of functions and the natural grading is $\mathbb{Z}_2$. Though [43] was devoted to the analogs of formal groups, the authors explained what the analog of a non-formal Lie group should be and gave two examples: in modern language, the general linear supergroup $\mathrm{GL}(n|m)$ and the diffeomorphism supergroup $\mathrm{Diff}(\mathbb{R}^{0|m})$.

(The notion of a Hopf algebra was discovered by Milnor, motivated by the study of cohomology operations. G.I. Kac, who should not be confused with V.G. Kac of Kac–Moody algebras, independently came to a close notion, which he called a 'ring group',

[24] Of course, the name came after the program was actually fulfilled.

[25] A possible confusion with the ordinary Lie algebras possessing a grading.

[26] As S.P. Novikov told to the writer of these words many years ago, much of Milnor and Moore's paper had been part of folklore before its publication.

working in representation theory. It was instrumental for his generalization of the Pontrjagin duality and the Tannaka–Krein duality. Works of G.I. Kac, who died untimely in 1978, anticipated the discovery of quantum groups; incidentally, before quantum groups no good examples of Hopf algebras that are neither commutative nor cocommutative were known. So the collaboration of Berezin and G.I. Kac on [43] was not accidental. Before [43], Nijenhuis came very close to the concept of a Lie supergroup in deformation theory. Nijenhuis used pairs consisting of a Lie superalgebra – of course, $\mathbb{Z}_2$ was $\mathbb{Z}$ *modulo* 2 – and a Lie group corresponding to its even part, which is an ordinary Lie algebra. Such pairs are equivalent to Lie supergroups and are nowadays sometimes referred to by the name 'Harish-Chandra pairs', borrowed from representation theory.)

After [17, 43], everything seems ready for the introduction of "spaces" for which the elements of $\mathbb{Z}_2$-graded algebras would be "functions". But in fact it required a few more years and some extra steps.

Such "spaces" remain implicit in paper [79] by G.I. Kac and A.I. Koronkevich, submitted shortly after [43], where a superanalog of Frobenius theorem in the language of differential forms was stated and proved.

A preliminary step was made in the setting of algebraic geometry. In paper [87], submitted in February 1973, Berezin's student D.A. Leites introduced a generalization of affine schemes (over a field) to the case of $\mathbb{Z}_2$-graded algebras. In particular, he introduced affine group schemes in this context and defined their Lie algebras (in the sense of [43], i.e., Lie superalgebras).

The missing ingredient – before differentiable supermanifolds would become possible – was transformations of variables mixing the ordinary variables with Grassmann generators. Berezin came to the idea of such transformations studying his integral: about the same time as [43] was written, he arrived at a formula for a general change of variables in the integral over a collection of anticommuting and ordinary variables. According to Minlos [93], the conjectural statement originally appeared in 1971 in a letter to G.I. Kac. It contained, in particular, the notion of a 'superdeterminant' (this name emerged only later [27]):

$$\operatorname{sdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \det(A - BD^{-1}C)\,(\det D)^{-1}\,. \tag{12}$$

Here the entries of A, D are even and the entries of B, C, odd. The change of variables formula reads (in the notation close to Berezin's own notation):

$$\int f(y,\eta)\,d\eta dy = \int f(y(x,\xi),\eta(x,\xi))\,J(x,\xi)\,d\xi dx\,, \tag{13}$$

where

$$J(x,\xi) = \operatorname{sdet}\begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial y}{\partial \xi} & \frac{\partial \eta}{\partial \xi} \end{pmatrix}\,. \tag{14}$$

The integral is extended to all values of the variables y and the function $f(y,\eta)$ must be vanishing at the infinity of y.

[27] Now the term 'Berezinian' and the notation Ber are universally adopted.

It is a curious fact that Berezin did not publish the definition of superdeterminant and the change of variables formula himself. Berezin suggested a proof of (13) as a problem to his student V.F. Pakhomov, in whose paper [103], submitted in December 1973, the above formulas first appeared in print[28].

Allowing changes of variables in the Berezin integral implied considering ordinary coordinates and Grassmann generators on an equal footing as generators of the algebra $C^\infty(\mathbb{R}^n) \otimes \Lambda(\mathbb{R}^m)$, denoted $\mathfrak{B}_{n,m}$ in [103] (in modern language it is $C^\infty(\mathbb{R}^{n|m})$). This was probably the final step towards supermanifolds.

An algebraic proof of the multiplicativity of "Berezin's function" (12) was given by D.A. Leites in a short note [88], submitted in May 1974.

5.4. Emergence of supersymmetric models and the explicit introduction of supermanifolds

The analysis in the previous subsection amply demonstrates that Berezin's program had been mainly fulfilled by himself and his collaborators by around 1973. The notion of a supermanifold was for them "in the air", though it had not appeared in the publications explicitly. The same can be said about supergroups. Still, according to Leites, Berezin felt reluctant to publish the definition of a supermanifold and was forced to do so only in order not to lose the priority. So what happened?

The momentum came again from theoretical physics and it was **supersymmetry**. Now this name is used very widely and sometimes outside of its precise original meaning, which is transformations of fields mixing the fermionic fields (usually describing 'matter') with bosonic fields (usually describing 'interaction').

In parallel with Berezin's work, the breakthrough was preparing in 1971–1974. Supersymmetry appeared, in the context of 'dual-resonance models' (later, string theory), in Ramond [104] and Neveu–Schwarz [100]; and, in the context of four-dimensional gauge theory, in Golfand–Likhtman [72], Volkov *et al.* [119, 120, 121] (who explicitly quoted Berezin and G.I. Kac [43]), and finally in Wess–Zumino [123, 124], whose work resulted in an explosion. Physicists started to look around for mathematical foundations of the new theory. Salam and Strathdee [105] were the first to formulate the concept of a 'superspace' on an operational level.

In such a context, Berezin was forced to act quickly. Berezin and Leites published [45]. This paper contained the definition of a supermanifold as a local ringed space modeled on open domains of $\mathbb{R}^p$ endowed with the $\mathbb{Z}_2$-graded algebra $C^\infty(\mathbb{R}^p) \otimes \Lambda(\mathbb{R}^q)$; the notions of morphisms of supermanifolds, subsupermanifolds and the direct products; the coordinate description by local charts and coordinate transformations; the notion of what we now call Berezin volume density and the construction of the Berezin integral over a supermanifold; specifying supermanifolds by equations in $\mathbb{R}^{n|m}$ and a conjecture that this may be possible for an arbitrary supermanifold (analog of Whitney's theorem); Lie supergroups (global) and their Lie superalgebras (so renamed from the 'Lie algebras' of [43]). Quite a

[28]Without any particular name and notation for the function sdet.

lot! Of course, the big work remained to elaborate the details and to make them available to the public.

It is time to say something about the terminology which involves the prefix 'super-': supermanifold, superalgebra, supergroup, superspace... In physical context, the term "supersymmetry" has a direct meaning as a "superior symmetry" exceeding other symmetries that keep bosons and fermions separate. In mathematics, the prefix 'super-' should be understood as an abbreviation from supersymmetry, – as having something to do with physical supersymmetries. Berezin himself did not overuse this prefix. It was done by others, and this made an unfortunate aftertaste. Nevertheless, such is the universally adopted terminology and there is no other choice but follow it.

5.5. Berezin's work on supermathematics in 1975–1980

There were several directions of Berezin's work after the introduction of supermanifolds.

The physical papers by Berezin and Marinov [46, 47] were devoted to the description of spin by means of supermathematics. These works are still on the borderline with the previous period: they do not use the word 'supermanifold' yet; the earlier paper [46] was submitted for publication just one month after [45]. There is an interesting historical material (mainly physical) in [47], in particular, references to Martin [90, 91]. Berezin came to this subject again in a joint paper with V.L. Golo [41] of 1980. It appeared only a few days before Berezin died; he could not see it published. One can also mention here the posthumous publication [42] on a chiral supersymmetric sigma-model.

A central topic of Berezin's research in 1975–1980 was the theory of Lie superalgebras and Lie supergroups, especially their representations and invariants. Berezin's methods were global, geometric and analytic (e.g., used tools such as invariant integral) rather than infinitesimal.

In a short article [27], Berezin studied the Lie supergroup $\mathrm{U}(p\,|q)$ and its unitary representations[29]. In particular, Berezin found the invariant integral and the radial parts of Laplace operators; he introduced "non-degenerate" or "typical" irreducible representations and found their characters.

The method sketched in [27] was elaborated and generalized in a series of five preprints [28, 29, 30, 31, 32] of 1977. They contain very interesting material; it would be fair to say that much of it has unfortunately remained not well understood yet. These preprints, originally published in a small number of copies, were later included in the expanded English edition[30] of [38].

Two joint papers of Berezin and V.S. Retakh [51, 52] of 1978 were devoted to classification of Lie superalgebras whose even part is semisimple. (The classification of simple Lie superalgebras over $\mathbb{C}$ was obtained by V.G. Kac around 1975,

[29] Berezin's own notation for this Lie supergroup was $U(p, q)$ and this should not lead to a confusion with the ordinary Lie group of pseudounitary matrices.

[30] We should be thankful to the editors for that. The quality of the English translation is sometimes poor: e.g., "resultant" is confused with "result", but it is still readable.

who interacted actively with Berezin at that time. Kac's classification remarkably brings forward superanalogs of classical matrix Lie algebras and the Lie algebras of vector fields, which is one more evidence for the "naturalness" of supergeometry.)

Berezin's last publication on representations of Lie supergroups was the paper with V.N. Tolstoy [56] dealing with a certain real compact form of the Lie supergroup OSp(1|2). (It appeared already after his death.)

Besides the Lie supergroups, Berezin actively worked on the general theory of supermanifolds. We should mention the expository preprint [33] and the survey paper [35] (both of 1979), and of course Berezin's work on a book on supermanifolds, which was incomplete at the time of his death. It was to appear only as a posthumous publication [38], compiled and edited by his friends such as A.A. Kirillov and V.P. Palamodov. (Palamodov, in particular, included there his own new results on the structure of supermanifolds.) The Russian version appeared in 1983 and the expanded English translation in 1987.

Three mathematical questions that attracted Berezin's attention are worth mentioning.

The first was the question about "points of supermanifolds". No doubt, the fact that "functions" on supermanifolds contain nilpotents makes it harder to understand them as compared to ordinary manifolds. Supermanifolds cannot be treated as sets with some structure. For example, the supermanifold $\mathbb{R}^{0|m}$, whose "algebra of functions" is the Grassmann algebra with m generators, set-theoretically consists of a single point; clearly, the structure of $\mathbb{R}^{0|m}$ cannot be attributed to this one-point set. At the same time, physicists working with "superspace" freely used "points" such as (x^a, ξ^μ) with the odd coordinates ξ^μ, whatever that could mean. Berezin's solution to that was in the introduction of an auxiliary Grassmann algebra $\mathfrak{G}(N)$ with a large or infinite number of generators N and in considering, for each supermanifold M, the ordinary manifold $M(N)$ (of large dimension) obtained by replacing abstract even and odd coordinates in the coordinate transformations for M by elements of $\mathfrak{G}(N)$ of corresponding parities. The manifold $M(N)$, by construction, has a special structure called by Berezin a *Grassmann-analytic* structure [31]. If N is large enough, the supermanifold M can be recovered from $M(N)$ taken with this structure. At the same time, the 'Grassmann-analytic manifold' $M(N)$ is a set-theoretic object and can be described by its points. Berezin used this idea widely, in particular for representations of Lie superalgebras and Lie supergroups, which he replaced by (in his terminology) their 'Grassmann envelopes'.

Berezin's idea about the manifolds $M(N)$ and 'Grassmann-analytic manifolds' in general contained the roots of several later developments (some probably independent of him). If one fixes a Grassmann algebra $\mathfrak{G}(N)$ and considers manifolds over it endowed with some class of 'Grassmann-analytic functions', there is

[31] This exactly means that the coordinate transformations on $M(N)$ are given by transformations of elements of the algebra $\mathfrak{G}(N)$ taken as whole quantities – like complex numbers instead of their real and imaginary parts taken separately. This is a particular case of what is called a 'manifold over an algebra'.

a temptation to forget about supermanifolds defined as ringed spaces altogether. Two versions of this idea were put forward, by B. DeWitt and by Alice Rogers, but in spite of all intuitive attractiveness, it was later found that its consistent development takes one back to the sheaf-theoretic approach to supermanifolds. Another option is not to fix $\mathfrak{G}(N)$, but consider the manifolds $M(N)$ as a functor of $\mathfrak{G}(N)$. Together they represent the original supermanifold M. This was suggested by A.S. Schwarz. It allows to consider objects more general than supermanifolds.

The second question concerned general classification of supermanifolds. It is obvious that the case in which the coordinate changes for a given supermanifold do not mix odd variables with the even variables and the odd variables transform linearly is the simplest and it corresponds to a vector bundle over an ordinary manifold. The question is how general this case is, i.e., whether it is always possible to reduce coordinate transformations to this simple form by a choice of atlas. The answer is, yes, – for smooth supermanifolds. This statement is often referred to as the Batchelor theorem after Marjorie Batchelor who proved it in 1979. No doubt that Berezin knew it independently: he mentions it in [33, 35]. As for complex-analytic supermanifolds, the answer is, no; there are obstructions. The corresponding theory and examples of "non-retractable" complex-analytic supermanifolds are due to V.P. Palamodov (a slightly different approach was developed by Yu.I. Manin).

Finally, the third question concerned integration on supermanifolds and differential forms. It is clear that Berezin's transformation law for the element of volume is different from what one gets for the differentials of coordinates defined in a straightforward way. So on supermanifolds, differential forms and integration theory seem to split. The problem was tackled by J.N. Bernstein and D.A. Leites, who introduced 'integral forms' [57], incorporating volume elements, as a replacement of differential forms for the purpose of integration and 'pseudodifferential forms' [58] as not necessarily polynomial expressions in differentials (which also opens way for integration).

Berezin in [36] introduced a further generalization of the Bernstein–Leites pseudodifferential forms, studied their duality transformations and sketched a Weil-type construction of characteristic classes. Berezin's aim was future application to super gauge theory. Paper [36] seems to be Berezin's last paper on supermanifolds. It is worth mentioning that [36] contained a construction very close to what is now known as the 'homological interpretation' of Berezinian.

5.6. Influence. Later developments

The influence of Berezin's work on supermathematics remains different in physics and in pure mathematics. Physicists have completely absorbed the idea of working with supermanifolds. For them, supergeometry is a tool on the same footing as tensor calculus: physicists use it without even noticing it. Unlike that, in pure mathematics, Berezin's ideas have spread far less widely. Supermanifolds for many remain something exotic (except for those directly working in supergeometry).

Quite characteristic is that representations of Lie superalgebras became a well-established area, but those working in it rarely consider Lie supergroups or turn to global methods used by Berezin. No doubt, the landscape would be quite different, had Berezin not died in 1980. However, the situation is slowly changing. "Supermethods" start to spread in differential geometry. Of course, this development is more significant in areas closer to or more influenced by physics. Two Fields medals awarded in 1990 and 1998, to E. Witten and M.L. Kontsevich, respectively, were related with works where supergeometry played a role. (Morse theory and differential forms, in the case of Witten, and deformation theory and quantization of Poisson manifolds, in the case of Kontsevich.)

One general trend worth mentioning is a certain shift from "supersymmetry" (roughly, transformations squaring to ordinary symmetries) to "BRST-symmetry" and "Q-manifolds" (where, roughly, there are transformations with square zero). A central role has been played here by the Batalin–Vilkovisky formalism in quantum field theory [3, 4, 5] and its modern generalizations. Geometrically, that means considering supermanifolds endowed with an odd symplectic structure and odd Laplacians on them. (The study of such geometry was pioneered by H.M. Khudaverdian [82], see also [83, 84].)

Another trend is the growth of importance of *graded manifolds* – not in the sense synonymous with supermanifolds as the usage in the early period sometimes was [32], – but meaning supermanifolds endowed with extra $\mathbb{Z}$- or $\mathbb{Z}_+$-grading, which in physics may be for example, 'ghost number'. If one recalls topologists' $\mathbb{Z}$-graded algebras and the replacement of $\mathbb{Z}$-grading by $\mathbb{Z}_2$-grading as a step in development of supermathematics, as described above, the reintroduction of $\mathbb{Z}$-gradings (but now as additional structure) completes the circle, but at a higher level.

We would like to finish this section by two interesting pieces of mathematics related with Berezin integration and Berezinian (superdeterminant), which were discovered after Berezin.

In the previous subsection, we considered the works on integration theory and differential forms by Bernstein–Leites and by Berezin [36]. In search of objects suitable for integration over (multidimensional) paths or surfaces in supermanifolds, a variational approach to "forms on supermanifolds" was developed (Th.Th. Voronov and A.V. Zorich, see [122]; building on earlier works by A.S. Schwarz and his students): analogs of forms were constructed as Lagrangians satisfying certain restrictions. An amazing fact discovered along these lines and not fully understood yet is the following link with integral geometry in the sense of Gelfand–Gindikin–Graev: the equation of the form [33]

$$\frac{\partial^2 f}{\partial w_a^i \partial w_b^j} + (-1)^{\tilde{\imath}\tilde{\jmath}+\tilde{a}(\tilde{\imath}+\tilde{\jmath})} \frac{\partial^2 f}{\partial w_a^j \partial w_b^i} = 0\,, \tag{15}$$

[32] In the Western literature, when the foundations of supermanifolds were thought to be not fully established yet, 'supermanifolds' were often used for DeWitt or Rogers's versions of manifolds over Grassmann algebras, while 'supermanifolds' in Berezin's sense were called 'graded manifolds'.

[33] The tilde over an index denotes the parity of the corresponding variable.

for a function of a rectangular supermatrix $\|w_a^i\|$, arises in the de Rham theory on supermanifolds as a condition replacing skew-symmetry and multilinearity (see [122]) and at the same time it is a generalization of 'hypergeometric equations' in the sense of Gelfand (the odd-odd part of (15) is the F. John equation arising in relation with the Radon transform).

Another beautiful development related with the notion of Berezinian is as follows. Th. Schmitt [107] discovered that the expansion of Berezinian leads to exterior powers:

$$\operatorname{Ber}(1+zA) = 1 + z\operatorname{str} A + z^2 \operatorname{str}\Lambda^2 A + \cdots .$$

Here A is an even supermatrix,

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

$\Lambda^k A$ stands for its action on the kth exterior power and str denotes the supertrace: $\operatorname{str} A = \operatorname{tr} A_{00} - \operatorname{tr} A_{11}$. As it was found in [85], by comparing the expansions of $\operatorname{Ber}(1+zA)$ at zero and at infinity one arrives at certain universal recurrence relations satisfied by the differences of the respective coefficients. In particular, for a $p\,|q \times p\,|q$ matrix, there are relations

$$\begin{vmatrix} c_k(A) & \dots & c_{k+q}(A) \\ \dots & \dots & \dots \\ c_{k+q}(A) & \dots & c_{k+2q}(A) \end{vmatrix} = 0\,,$$

where $c_k(A) = \operatorname{str}\Lambda^k A$, satisfied for all $k > p-q$. (This replaces the vanishing of the kth exterior powers for an n-dimensional space with $k > n$). Similar relations hold in the Grothendieck ring of a general linear supergroup, and there is a formula

$$\operatorname{Ber} A = \frac{|\,c_{p-q}(A)\dots c_p(A)\,|_{q+1}}{|\,c_{p-q+2}(A)\dots c_{p+1}(A)\,|_q}\,,$$

with Hankel's determinants at the top and at the bottom, expressing Berezinian as the ratio of polynomial invariants [34].

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[34] In the definition, $\operatorname{Ber} A = \det(A_{00} - A_{01}A_{11}^{-1}A_{10})\det A_{11}^{-1}$, neither the numerator nor the denominator of the fraction are invariant.

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Some Non-standard Examples of Coherent States and Quantization

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Abstract. We look at certain non-standard constructions of coherent states, viz., over matrix domains, on quaternionic Hilbert spaces and C*-Hilbert modules and their possible use in quantization. In particular we look at families of coherent states built over Cuntz algebras and suggest applications to non-commutative spaces. The present considerations might also suggest an extension of Berezin-Toeplitz and coherent state quantization to quaternionic Hilbert spaces and Hilbert modules.

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1. Standard coherent states

Coherent states are a much used concept, both physically and mathematically. Generically, they are obtained from a reproducing kernel subspace (see, for example, [1]) of an L^2-space,

$$\mathfrak{H}_K \subset \mathfrak{H} = L^2(X, \mu),$$

where μ is a finite measure on the Borel σ-field of a locally compact topological space X. If

$$\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots$$

is any orthonormal basis of $\mathfrak{H}_K$, then the reproducing kernel is given by

$$K(x, y) = \sum_k \Phi_k(x)\overline{\Phi_k(y)}\ . \tag{1}$$

Using this fact and taking another Hilbert space $\mathfrak{K}$ of the same dimension as that of $\mathfrak{H}_K$, the non-normalized coherent states are defined as

$$|\, x\rangle = \sum_k \psi_k\ \overline{\Phi_k(x)}, \tag{2}$$

where $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ is an orthonormal basis of $\mathfrak{K}$.

It is then easy to verify that

$$\langle x \mid y\rangle = K(x,y) \quad \text{and} \quad \int_X \mid x\rangle\langle x \mid \; d\mu(x) = I_{\mathfrak{K}} \; , \tag{3}$$

the integral converging in the weak operator topology. If, furthermore,

$$K(x,x) = \sum_k |\Phi_k(x)|^2 := \mathcal{N}(x) > 0,$$

for all $x \in X$, normalized CS can be defined as:

$$\widehat{\mid x\rangle} = \mathcal{N}(x)^{-\frac{1}{2}} \mid x\rangle \; ,$$

which then satisfy the conditions,

$$\|\widehat{\mid x\rangle}\| = 1 \quad \text{and} \quad \int_X \widehat{\mid x\rangle}\widehat{\langle x \mid} \mathcal{N}(x) \; d\mu(x) = I_{\mathfrak{K}} \; .$$

These are the physical coherent states.

Berezin-Toeplitz quantization or, coherent state quantization, of functions f on the space X is given by the operator association (see, for example, [2] and references cited therein),

$$f \longmapsto \hat{f} = \int_X f(x)|x\rangle\langle x| \; d\mu(x) \; , \tag{4}$$

provided the integral exists in some appropriate sense.

In view of their usefulness and interest in various areas of physics and mathematics, it is natural to look for generalizations of the above concept of coherent states.

One such possibility is to construct analogous objects on a Hilbert C^*-module, which is analogous to a Hilbert space, but has an inner product taking values in a C^*-algebra. We shall call the resulting vectors module-valued coherent states (MVCS). In simple terms, we shall replace both the set of functions $\Phi_k(x)$ and the vectors ψ_k, in the definition of coherent states in (2) by elements of Hilbert modules. Another possibility for generalization could be to construct coherent states on quaternionic Hilbert spaces.

Since the field of complex numbers $\mathbb{C}$ is trivially a C^*-algebra, coherent states on Hilbert spaces are special cases of MVCS.

2. Module-valued coherent states

The discussion of this section is based mainly on [3]. Consider two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and a Hilbert C^*-correspondence $\mathbf{E}$ from $\mathcal{A}$ to $\mathcal{B}$. This means that $\mathbf{E}$ is a Hilbert C^*-module over $\mathcal{B}$, with a left action from $\mathcal{A}$, i.e., there is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{L}(\mathbf{E})$, the bounded adjointable operators on $\mathbf{E}$. Let (X,μ) be a finite measure space and consider the set of functions,

$$\mathbb{F} = \{F : X \longmapsto \mathbf{E} \mid F \;\text{ is a strongly measurable function}\} \; .$$

Then clearly, for any two F, G in $\mathbb{F}$, $x \longmapsto \langle F(x) \mid G(x)\rangle_{\mathbf{E}}$ is a strongly measurable function. Let

$$\mathfrak{H} = \{F \in \mathbb{F} \mid \text{ the function} \langle F(x) \mid F(x)\rangle \text{ is Bochner integrable}\} . \tag{5}$$

Given a strongly measurable function F, a necessary and sufficient condition for $\langle F(x) \mid F(x)\rangle$ to be Bochner integrable is that

$$\int_X \|\langle F(x) \mid F(x)\rangle_{\mathbf{E}}\|_{\mathcal{B}}\, d\mu(x) < \infty .$$

This immediately shows that $\mathfrak{H}$ is a complex vector space. Also, $\mathfrak{H}$ is an inner product module over $\mathcal{B}$, where the right multiplication and the inner product respectively are

$$(F \cdot b)(x) = F(x)b \text{ for all } b \in \mathcal{B}, \ \ \langle F \mid G\rangle_{\mathfrak{H}} = \int_X \langle F(x) \mid G(x)\rangle_{\mathbf{E}}\, d\mu(x).$$

Its completion in the resulting norm $\|F\|_{\mathfrak{H}} = \|\langle F \mid \ F\rangle_{\mathfrak{H}}\|_{\mathcal{B}}^{\frac{1}{2}}$ is a Hilbert C^*-module over $\mathcal{B}$ and can be identified with $L^2(X) \otimes \mathbf{E}$. There is a natural left action of $\mathcal{A}$ on $\mathfrak{H}$ because $\mathbf{E}$ is an $\mathcal{A} - \mathcal{B}$ correspondence.

For $e \in \mathbf{E}$, we define the map $\langle e| : \mathbf{E} \longrightarrow \mathcal{B}$, by

$$\langle e|(f) = \langle e \mid f\rangle_{\mathbf{E}} , \quad f \in \mathbf{E} .$$

This is an adjointable map. We shall denote its adjoint by $|e\rangle$. Then $|e\rangle : \mathcal{B} \longrightarrow \mathbf{E}$ has the action

$$|e\rangle(b) = eb , \quad b \in \mathcal{B} ,$$

so that for $e_1, e_2 \in \mathbf{E}$,

$$|e_1\rangle\langle e_2|(f) = e_1\langle e_2 \mid f\rangle_{\mathbf{E}} . \tag{6}$$

Thus formally, one may use the standard bra-ket notation for Hilbert modules as one does for Hilbert spaces.

Let us choose a set of vectors

$$F_0, F_1, \ldots, F_n, \ldots ,$$

(finite or infinite) in the function space $\mathfrak{H}$, which are pointwise defined (for all $x \in X$) and which satisfy the orthogonality relations,

$$\int_X \mid F_k(x)\rangle\langle F_\ell(x) \mid \ d\mu(x) = I_{\mathbf{E}}\, \delta_{k\ell} . \tag{7}$$

We now introduce module-valued coherent states for two separate situations, highlighting the fact that a Hilbert C^*-module is a generalization of both a Hilbert space and a C^*-algebra. The resulting MVCS depend on an auxiliary object $\mathbf{G}$, which in the first instance is a Hilbert space and in the second, the Cuntz algebras $\mathcal{O}_n$ or $\mathcal{O}_\infty$.

To proceed with the first construction of MVCS let $\mathbf{G}$ be a Hilbert space of the same dimension as the cardinality of the F_k. In $\mathbf{G}$ we choose an orthonormal basis, $\phi_0, \phi_1, \ldots, \phi_n, \ldots$. Let $\mathbf{H} = \mathbf{E} \otimes \mathbf{G}$ denote the exterior tensor product of $\mathbf{E}$ and $\mathbf{G}$, which is then itself a Hilbert module over $\mathcal{B}$.

For each $x \in X$ and co-isometry $a \in \mathcal{A}$ (i.e., $aa^* = \mathrm{id}_{\mathcal{A}}$), we define the vectors,

$$| \, x, a\rangle = \sum_k aF_k(x) \otimes \phi_k \in \mathbf{H} \ , \tag{8}$$

assuming of course that the sum converges in the norm of $\mathbf{H}$. We call these vectors (non-normalized) module-valued coherent states (MVCS).

Proposition 2.1. *The MVCS in* (8) *satisfy the resolution of the identity,*

$$\int_X | \, x, a\rangle\langle x, a \, | \ \ d\mu(x) = I_{\mathbf{H}}, \tag{9}$$

the integral converging in the sense that for any two $h_1, h_2 \in \mathbf{H}$,

$$\int_X \langle h_1 \mid x, a\rangle_{\mathbf{H}} \langle x, a \mid h_2\rangle_{\mathbf{H}} \, d\mu(x) = \langle h_1 \mid h_2\rangle_{\mathbf{H}} \ ,$$

as a Bochner integral.

This construction may easily be modified to obtain normalized MVCS under certain conditions. For that, we fix a notation for a certain positive element of $\mathcal{B}$. Let

$$\mathcal{N}(x, a) := \langle x, a \mid x, a\rangle_{\mathbf{H}} = \sum_k \langle F_k(x) \mid a^* aF_k(x)\rangle_{\mathbf{E}} \ . \tag{10}$$

Proposition 2.2. *If $\phi_1, \phi_2, \ldots$ is an orthonormal basis for $\mathbf{G}$ and a is a unitary element of $\mathcal{A}$ and $\mathcal{N}(x, id_{\mathcal{A}})$ is invertible, then the MVCS constructed above can be normalized, i.e., we can construct MVCS $\widehat{| \, x, a\rangle} = | \, x, a\rangle \otimes \mathcal{N}(x, id_{\mathcal{A}})^{-\frac{1}{2}}$ which along with* (7) *also satisfy*

$$\langle \widehat{x, a} \mid \widehat{x, a}\rangle = id_{\mathcal{B}} \otimes id_{\mathcal{C}} \ . \tag{11}$$

The well-known vector coherent states [4, 5] (or multi-component coherent states), used in nuclear and atomic physics, can all be obtained from module-valued coherent states using the above construction. Furthermore, one can define adjointable operators on the Hilbert module $\mathbf{H}$ following a Berezin-Toeplitz type prescription as in (4):

$$f \longrightarrow \widehat{f} = \int_X f(x) |x, \mathrm{id}_{\mathcal{A}}\rangle\langle x, \mathrm{id}_{\mathcal{A}}| \ d\mu(x) \ ,$$

and study the resulting quantization problem.

3. MVCS from certain Cuntz algebras

We now construct MVCS using the notion of Cuntz algebras [6] (see also [7]). Let $S_1, S_2, \ldots$ be isometries on a complex separable Hilbert space $\mathcal{K}$ (necessarily infinite-dimensional) such that

$$\sum_{j=1}^{\infty} S_j S_j^* = I_{\mathcal{K}}$$

where the sum converges in the strong operator topology of $\mathcal{B}(\mathcal{K})$. Multiplying both sides by S_i^*, we get

$$S_i^* + S_i^* \sum_{j \neq i} S_j S_j^* = S_i^*$$

so that

$$S_i^* \sum_{j \neq i} S_j S_j^* = 0 \ .$$

But $\sum_{j \neq i} S_j S_j^*$ is the projection onto the closure of the span of the ranges of S_j for $j \neq i$. So the range of S_i is orthogonal to the range of S_j for all $j \neq i$. This is a representation of the Cuntz algebra $\mathcal{O}_\infty$ with infinitely many generators.

We take $\mathbf{G}$ to be the C^*-algebra generated by the isometries $S_1, S_2, \ldots$. The coherent states are defined as

$$|x, a\rangle = \left(\sum_{k=1}^{\infty} a \cdot F_k(x) \otimes S_k \right) (\boldsymbol{\mathcal{N}}(x)^{-1/2} \otimes I). \tag{12}$$

An explicit example of a Cuntz algebra is as follows. Let

$$\omega : \mathbb{N}^{>0} \longrightarrow \mathbb{N}^{>0} \times \mathbb{N}^{>0}$$

be a bijection ($\mathbb{N}^{>0}$ denoting the set of positive integers). Consider a Hilbert space $\mathfrak{H}$ and let $\{\phi_n\}_{n \in \mathbb{N}^{>0}}$ be an orthonormal basis of it. Writing $\omega(n) = (k, \ell)$ we define a re-transcription of this basis in the manner

$$\psi_{k\ell} := \phi_n = \psi_{\omega(n)} \ , \qquad k, n, \ell \in \mathbb{N}^{>0} \ . \tag{13}$$

The C^*-algebra $\mathcal{O}_\infty$, generated by these isometries, is then a Cuntz algebra.

The MVCS obtained using these S_k in (12) have an immediate physical application. We consider the non-normalized version (with a set to the unit element of $\mathcal{A}$),

$$|x\rangle = \sum_{k=1}^{\infty} F_k(x) \otimes S_k.$$

Let

$$X = \mathbb{C} \qquad \text{and} \qquad \mathbf{E} = L^2\left(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\, dx\, dy\right), \ \ z = \frac{1}{\sqrt{2}}(x + iy) \ ,$$

and let $F_k : \mathbb{C} \longrightarrow \mathbb{C}$ be the functions,

$$F_k(z) = \frac{z^{k-1}}{\sqrt{(k-1)!}} \ , \qquad k = 1, 2, 3, \ldots \ .$$

Next let $\psi_{k\ell}$ be the complex Hermite polynomials,

$$\psi_{k\ell}(\overline{z}, z) = \frac{(-1)^{n+k-2}}{\sqrt{(\ell-1)!(k-1)!}}\ e^{|z|^2} \partial_{\overline{z}}^{\ell-1} \partial_z^{k-1}\ e^{-|z|^2} \ , \qquad k, \ell = 1, 2, 3, \ldots \ , \tag{14}$$

which form an orthonormal basis of $L^2\big(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\ dx\ dy\big)$. The module-valued coherent states now become

$$|z\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \ . \tag{15}$$

Let ϕ_n be as in (13), consider the vectors

$$\xi_{\overline{z}',\, n} = \frac{\overline{z}'^{n-1}}{\sqrt{(n-1)!}} \phi_n \ .$$

Then the vectors (in $L^2(\mathbb{C},\ \frac{e^{-|z|^2}}{2\pi}\ dx\ dy)$),

$$|z, \overline{z}', n\rangle = \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!}} S_k \xi_{\overline{z}',\, n} = \overline{z}'^{n-1} \sum_{k=1}^{\infty} \frac{z^{k-1}}{\sqrt{(k-1)!\,(n-1)!}}\ \psi_{kn}\ , \tag{16}$$

($\ell = 1, 2, 3, \ldots, \infty$,) are just the non-normalized versions of the infinite component vector CS found in [5] and associated to the energy levels (the so-called Landau levels) of an electron in a constant magnetic field.

4. Matrix-valued and quaternionic MVCS

In [4] analytic vector coherent states, built using powers of matrices from $\mathcal{M}_N(\mathbb{C})$, were defined:

$$|\,\mathfrak{Z}, i\rangle = \sum_k \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \chi^i \otimes \mathbf{\Phi}_k\ , \qquad \mathfrak{Z} \in \mathcal{M}_N(\mathbb{C})\ , \tag{17}$$

where the c_k are the numbers,

$$c_k = \frac{1}{(k+1)(k+2)} \left[\prod_{j=1}^{k+1} (N+j) - \prod_{j=1}^{k+1} (N-j) \right], \qquad k = 0, 1, 2, \ldots,$$

Let $z_{ij},\ i,j = 1, 2, \ldots, N$ be the matrix elements of $\mathfrak{Z}$. Then, writing

$$F_k(\mathfrak{Z}) = \frac{\mathfrak{Z}^k}{\sqrt{c_k}} \quad \text{and} \quad z_{ij} = x_{ij} + i y_{ij}\ ,$$

it can be shown that,

$$\int_{\mathcal{M}_N(\mathbb{C})} F_k(\mathfrak{Z}) F_\ell(\mathfrak{Z})^*\ d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \delta_{k\ell} \mathbb{I}_N\ ,$$

where

$$d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \frac{e^{-\mathrm{Tr}[\mathfrak{Z}^*\mathfrak{Z}]}}{(2\pi)^N} \prod_{i,j=1}^{N} dx_{ij}\ dy_{ij}\ .$$

Using this fact, it is easy to prove the resolution of identity,

$$\sum_{i=1}^{N} \int_{\mathcal{M}_N(\mathbb{C})} |\,\mathfrak{Z}, i\rangle\langle \mathfrak{Z}, i\,|\ \ d\mu(\mathfrak{Z}, \mathfrak{Z}^*) = \mathbb{I}_N \otimes I_{\mathfrak{H}_{\mathbf{K}}}\ .$$

To construct the related MVCS, we take $\mathbf{E} = \mathcal{B} = \mathcal{M}_N(\mathbb{C})$. The module $\mathfrak{H}$, containing the functions F_k, then consists of functions from $\mathcal{M}_N(\mathbb{C})$ to itself. Considering $\mathfrak{H}_{\mathbf{K}}$ as a module over $\mathbb{C}$, we may define MVCS in $\mathbf{H} = \mathcal{M}_N(\mathbb{C}) \otimes \mathfrak{H}_{\mathbf{K}}$ as

$$\mid \mathfrak{z}, a\rangle = \sum_k aF_k(\mathfrak{z}) \otimes \mathbf{\Phi}_k = \sum_k a\frac{\mathfrak{z}^k}{\sqrt{c_k}} \otimes \mathbf{\Phi}_k \ , \tag{18}$$

where a is a unitary element in $\mathcal{M}_N(\mathbb{C})$. These then satisfy the resolution of the identity,

$$\int_{\mathcal{M}_N(\mathbb{C})} \mid \mathfrak{z}, a\rangle\langle \mathfrak{z}, a \mid \ d\mu(\mathfrak{z}, \mathfrak{z}^*) = I_{\mathbf{H}} \ . \tag{19}$$

In the particular case when $N = 2$ the set $\mathcal{M}_N(\mathbb{C})$, of all complex 2×2 matrices, can be identified with the space of complex quaternions. The resulting MVCS may then be called complex quaternionic MVCS.

Although a Hilbert space over the quaternions is not a Hilbert module, we may still build coherent states in such a space using the above construction on Hilbert modules. Such coherent states also have interesting physical applications [8]. Suppose that $\mathfrak{H}_{\text{quat}}$ is a Hilbert space over the quaternions. (Multiplication by elements of $\mathbb{H}$ from the right is assumed, i.e., if $\mathbf{\Phi} \in \mathfrak{H}_{\text{quat}}$ and $\mathfrak{q} \in \mathbb{H}$, then $\mathbf{\Phi}\mathfrak{q} \in \mathfrak{H}_{\text{quat}}$). The well-known canonical coherent states [1] may then be readily generalized to quaternionic coherent states over $\mathfrak{H}_{\text{quat}}$. Indeed take an orthonormal basis $\{\mathbf{\Psi}_n^{\text{quat}}\}_{n=0}^{\infty}$ in $\mathfrak{H}_{\text{quat}}$ and define the vectors

$$\mid \mathfrak{q}\rangle = e^{-\frac{r^2}{2}} \sum_{n=0}^{\infty} \mathbf{\Psi}_n^{\text{quat}} \frac{\mathfrak{q}^n}{\sqrt{n!}} \in \mathfrak{H}_{\text{quat}}, \qquad \mathfrak{q} \in \mathbb{H}, \qquad \langle \mathfrak{q} \mid \mathfrak{q}\rangle_{\mathfrak{H}_{\text{quat}}} = \mathbb{I}_2 \ . \tag{20}$$

They satisfy the resolution of the identity,

$$\int_{\mathbb{H}} \mid \mathfrak{q}\rangle\langle \mathfrak{q} \mid \ d\nu(\mathfrak{q}, \mathfrak{q}^\dagger) = I_{\mathfrak{H}_{\text{quat}}} \ , \qquad d\nu(\mathfrak{q}, \mathfrak{q}^\dagger) = \frac{1}{4\pi^2}\, r dr \ d\xi \ \sin\theta d\theta \ d\phi \ . \tag{21}$$

In [8] these coherent states were obtained using a group theoretical argument. Here they appear as a special case of our more general construction.

5. Some possible applications

We end this discussion by mentioning some possible applications of the above general constructions of non-standard families of coherent states.

- Coherent states are naturally associated to positive definite kernels [1], coming from the reproducing kernel Hilbert spaces used to build them. It would be interesting to study such kernels for the MVCS and the coherent states on quaternionic Hilbert spaces. Then there would also be related positive operator-valued measures and a Naimark type dilation theorem. One could also study subnormal operators in this context.
- As already mentioned, a Berezin-Toeplitz type quantization on Hilbert modules would be a natural problem to study.

- Module-valued coherent states have been used to define localization on non-commutative spaces [3], which is another direction for further investigation. Indeed, it is in this direction, where standard quantum mechanics might not be readily applicable, that we see greater possibility of application of this general concept.

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Classical and Quantum Evolution on the Siegel-Jacobi Manifolds

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Abstract. Under a homogeneous Kähler transform, the Kähler two-form on the Siegel-Jacobi disk $\mathcal{D}_1^J$ (upper half-plane $\mathcal{X}_1^J$) splits into the sum of the Kähler two-form on $\mathbb{C}$ and Kähler two-form on the Siegel disk $\mathcal{D}_1$ (respectively, the Siegel upper half-plane $\mathcal{X}_1$). Similar considerations are presented in the case of the Jacobi group acting on Sigel-Jacobi ball $\mathcal{D}_n^J$ and Siegel-Jacobi space $\mathcal{X}_n^J$. We describe the classical and quantum evolution on the Siegel-Jacobi manifolds determined by a linear Hamiltonian in the generators of the Jacobi group.

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1. Introduction

We consider the Jacobi group $G_n^J = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$ [1, 2, 3, 4, 5], where H_n denotes the Heisenberg group. G_n^J acts transitively on the Siegel-Jacobi ball $\mathcal{D}_n^J = H_n/\mathbb{R} \times \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}/\mathrm{U}(n) = \mathbb{C}^n \times \mathcal{D}_n$, where the Siegel ball is realized as $\mathcal{D}_n := \{W \in M(n,\mathbb{C}) | W = W^t,\ 1 - W\bar{W} > 0\}$. $M(n,\mathbb{F})$ denotes the set of $n \times n$ matrices with entries in the field $\mathbb{F}$. We reserve the name of Siegel-Jacobi disk for $\mathcal{D}_1^J$. We have attached coherent states [6] to G_n^J, based on $\mathcal{D}_n^J$ [7]. The case $G_1^J = H_1 \rtimes \mathrm{SU}(1,1)$ was considered in [8, 9]. Previously, similar constructions were done in [10],[11]. The *squeezed states* [12, 13, 14, 15] in quantum optics [16, 17, 18] can be constructed as coherent states attached to the Jacobi group. We have determined the G_n^J-invariant Kähler two-form ω_n on $\mathcal{D}_n^J$ [7]. ω_n is a particular case of the Kähler two-form obtained in [19], written in a condensed form. In [20] we have determined the Kähler invariant two-form ω'_n on the Siegel-Jacobi space $\mathcal{X}_n^J := \mathcal{X}_n \times \mathbb{R}^{2n}$, also studied in detail by Yang in a larger context coming from an extended Heisenberg group [21]. ω'_n generalizes the expression of the Kähler two-form w_1' obtained by Kähler and Berndt [22, 5]. $\mathcal{X}_n$ denotes the Siegel upper half-plane (of order n).

In this note we report on the homogeneous Kähler transform which splits ω_n into the sum of the group-invariant Kähler two-forms on $\mathcal{D}_n$ and the one on $\mathbb{C}^n$, and we interpret this change of coordinates in the context of the celebrated Gindikin-Vinberg [23] fundamental conjecture for homogeneous Kähler manifold [24] (Proposition 7). The case of $\mathcal{D}_1^J$ is treated separately because is simpler to be presented (cf. Proposition 3); more details are given in [25].

We also investigate the motion on the Siegel-Jacobi manifolds $\mathcal{D}_n^J$ and $\mathcal{X}_n^J$ generated by a hermitian Hamiltonian $\boldsymbol{H}$ linear in the generators of G_n^J. Following Berezin's *dequantization* recipe [26, 27], we attache to $\boldsymbol{H}$ its covariant symbol $\mathcal{H}$. Using a technique developed in [28, 29] for a linear Hamiltonian in the generators of a Lie group G acting on a Kähler homogeneous manifold G/H in the case when the generators admit a realization as first-order holomorphic differential operators with polynomial coefficients (Proposition 4), we write down the equations of motion on the Siegel-Jacobi manifolds. Under the homogeneous Kähler transform which splits the Kähler two-forms on the Siegel-Jacobi manifolds, the equations of motion on $\mathbb{C}^n$ decouples of those on Siegel-Jacobi ball and space. In the case $n = 1$ the solution of the differential equations of motion are written down explicitly in the autonomous case.

In Section 2 we recall some notation and results from [8]. Proposition 3 establishes the homogeneous Kähler diffeomorphism for the Siegel-Jacobi manifolds in the case $n = 1$. The classical and quantum evolution on $\mathcal{D}_1^J$ and $\mathcal{X}_1^J$ are summarized in Proposition 6 of Section 4. In the case of G_n^J we get first-order matrix differential equations of motion on the Siegel-Jacobi manifolds. The motions on $\mathcal{D}_n$ and $\mathcal{X}_n$ are described by matrix Riccati equations, solved in [28, 29], while the decoupled equations of motion on $\mathbb{C}^n$ are complex first-order linear differential equations. More details will be published later.

2. Coherent states attached to the Jacobi group G_1^J

The Jacobi algebra is the semi-direct sum $\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1)$ [8, 9], where the Heisenberg algebra $\mathfrak{h}_1$ is generated by the boson operators $a, a^\dagger$ and 1, $[a, a^\dagger] = 1$, $\mathfrak{su}(1,1)$ is generated by $K_{\pm,0}$, and

$$[a, K_+] = a^\dagger,\ \left[K_-, a^\dagger\right] = a,\ \left[K_+, a^\dagger\right] = [K_-, a] = 0,$$
$$\left[K_0, a^\dagger\right] = \frac{1}{2}a^\dagger,\ [K_0, a] = -\frac{1}{2}a.$$

For $X \in \mathfrak{g}$, we denote $\boldsymbol{X} = d\pi(X)$, where π is unitary irreducible representation of a Lie group G with Lie algebra $\mathfrak{g}$. We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$\boldsymbol{a}e_0 = 0,\ \boldsymbol{K}_-e_0 = 0,\ \boldsymbol{K}_0e_0 = ke_0;\ k > 0, 2k = 2, 3, \ldots, \tag{1}$$

and k indexes the positive discrete series representations D_k^+ of $\mathrm{SU}(1,1)$ [30].

Perelomov's coherent state vectors are vectors in the Hilbert space of the representation of the group G^J_1, based on Siegel-Jacobi disk $\mathcal{D}^J_1$:

$$e_{z,w} := \mathrm{e}^{z\boldsymbol{a}^\dagger + w\boldsymbol{K}_+}e_0,\ z,w\in\mathbb{C},\ |w|<1. \tag{2}$$

We consider the *squeezed* CS vector $\Psi_{\alpha,w} := D(\alpha)S(w)e_0$ [13], where

$$D(\alpha) = \exp(\alpha\boldsymbol{a}^\dagger - \bar{\alpha}\boldsymbol{a}) = \exp\left(-\frac{1}{2}|\alpha|^2\right)\exp(\alpha\boldsymbol{a}^\dagger)\exp(-\bar{\alpha}\boldsymbol{a}),$$
$$\underline{S}(z) = \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-) = \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-),$$
$$w = \frac{z}{|z|}\tanh(|z|), \eta = \log(1-w\bar{w}).$$

Proposition 1. *The kernel* $K = (e_{\bar{z},\bar{w}}, e_{\bar{z},\bar{w}}) : \mathcal{D}^J_1\times\bar{\mathcal{D}}^J_1 \to \mathbb{C}$ *is*

$$K = (1-w\bar{w})^{-2k}\exp\frac{2z\bar{z} + z^2\bar{w} + \bar{z}^2 w}{2(1-w\bar{w})},\ z,w\in\mathbb{C},\ |w|<1. \tag{3}$$

The normalized squeezed state vector and the un-normalized Perelomov's coherent state vector are related by the relation

$$\Psi_{\alpha,w} = (1-w\bar{w})^k\exp\left(-\frac{\bar{\alpha}}{2}z\right)e_{z,w},\ z = \alpha - w\bar{\alpha}. \tag{4}$$

The composition law in the Jacobi group $G^J_1 := HW \rtimes SU(1,1)$ *is*

$$(g_1,\alpha_1,t_1)\circ(g_2,\alpha_2,t_2) = (g_1\circ g_2, g_2^{-1}\cdot\alpha_1 + \alpha_2, t_1 + t_2 + \Im(g_2^{-1}\cdot\alpha_1\bar{\alpha}_2).$$

The action of $(g,\alpha)\in G^J_1$ *on* $(z,w)\in\mathcal{D}^J_1$ *is given by*

$$z_1 = \frac{\alpha - \bar{\alpha}w + z}{\bar{q}w+\bar{p}};\ w_1 = \frac{pw+q}{\bar{q}w+\bar{p}},\ g = \begin{pmatrix} p & q\\ \bar{q} & \bar{p}\end{pmatrix}\in SU(1,1). \tag{5}$$

The Kähler two-form ω_1 *on* $\mathcal{D}^J_1$, G^J_1*-invariant to the action* (5), *is*

$$-\mathrm{i}\omega_1 = \frac{2k}{(1-w\bar{w})^2}dw\wedge d\bar{w} + \frac{A\wedge\bar{A}}{1-w\bar{w}}, A = dz + \bar{\eta}dw, \eta = \frac{z+\bar{z}w}{1-w\bar{w}}. \tag{6}$$

Let us consider the real Jacobi group $G^J_1(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})\ltimes\mathbb{R}^2$ acting on the Siegel-Jacobi upper half-plane $\mathcal{X}^J_1 := \mathcal{X}_1\times\mathbb{C}$, where $\mathcal{X}_1$ is the Siegel upper half-plane $\mathcal{X}_1 := \{v\in\mathbb{C}|\Im(v)>0\}$ [8],[22]. If $M\in\mathrm{SL}_2(\mathbb{R})$ then $M_*\in\mathrm{SU}(1,1)$, where

$$M_* = C^{-1}MC,\ C = \begin{pmatrix}\mathrm{i} & \mathrm{i}\\ -1 & 1\end{pmatrix}. \tag{7}$$

Remark 2. We have the biholomorphic map: $\mathcal{X}^J_1\to\mathcal{D}^J_1$

$$w = \frac{v-\mathrm{i}}{v+\mathrm{i}};\ z = \frac{2\mathrm{i}u}{v+\mathrm{i}},\ w\in\mathcal{D}_1,\ v\in\mathcal{X}_1,\ z,u\in\mathbb{C}. \tag{8}$$

Under the partial Cayley transform (8), ω_1 (6) becomes

$$-\mathrm{i}\,\omega_1' = -\frac{2k}{(\bar{v}-v)^2}dv\wedge d\bar{v} + \frac{2}{\mathrm{i}(\bar{v}-v)}G\wedge\bar{G},\ G = du - \frac{u-\bar{u}}{v-\bar{v}}dv. \tag{9}$$

ω'_1 is Kähler homogeneous under the action of $G^J_1(\mathbb{R})$ on $\mathcal{X}^J_1$:

$$v_1 = \frac{av+b}{cv+d}, u_1 = \frac{u+nv+m}{cv+d}, h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \alpha = m + \mathrm{i}n, \tag{10}$$

where the matrices g in (5) and h in (10) are related by (7).

3. The homogeneous Kähler diffeomorphisms for $\mathcal{D}^J_1$, $\mathcal{X}^J_1$

In the formulation of Dorfmeister and Nakajima [24], the fundamental conjecture for homogeneous Kähler manifolds (Gindikin -Vinberg [23]) essentially asserts that: *every homogeneous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogeneous manifold (generalized flag manifold), a homogeneous bounded domain, and $\mathbb{C}^n/\Gamma$, where Γ denotes a discrete subgroup of translations of $\mathbb{C}^n$*. In our case, we have:

Proposition 3. *Let us consider the Kähler two-form ω_1 (6), G^J_1-invariant under the action (5) of G^J_1 on the homogeneous Kähler Siegel-Jacobi disk $\mathcal{D}^J_1$. We have the homogeneous Kähler diffeomorphism*

$$FC: \ (\mathcal{D}^J_1, \omega_1) \rightarrow (\mathcal{D}_1 \times \mathbb{C}, \omega_0) = (\mathcal{D}_1, \omega_{\mathcal{D}_1}) \otimes (\mathbb{C}, \omega_{\mathbb{C}}), \ \omega_0 = FC(\omega_1),$$

$$FC: \ z = \eta - w\bar{\eta}, \ FC^{-1}: \ \eta = \frac{z + w\bar{z}}{1 - |w|^2}, \tag{11}$$

$$\omega_0 = \omega_{\mathcal{D}_1} + \omega_{\mathbb{C}}; \ -\mathrm{i}\omega_{\mathcal{D}_1} = \frac{2k}{(1 - w\bar{w})^2} dw \wedge d\bar{w}, \ -\mathrm{i}\omega_{\mathbb{C}} = d\eta \wedge d\bar{\eta}. \tag{12}$$

The Kähler two-form (12) is invariant at the action $(g, \alpha) \cdot (\eta, w) \rightarrow (\eta_1, w_1)$ of G^J_1 on $\mathbb{C} \times \mathcal{D}_1$, where

$$\eta_1 = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}), \ w_1 = \frac{aw + b}{\bar{b}w + \bar{a}}, \ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1). \tag{13}$$

We have also the homogeneous Kähler diffeomorphism

$$FC_1: (\mathcal{X}^J_1, \omega'_1) \rightarrow (\mathcal{X}_1 \times \mathbb{C}, \omega'_0) = (\mathcal{X}_1, \omega_{\mathcal{X}_1}) \times (\mathbb{C}, \omega_{\mathbb{C}}), \ \omega'_0 = FC_1(\omega'_1),$$

$$FC_1: 2\mathrm{i}u = (v + \mathrm{i})\eta - (\bar{v} - \mathrm{i})\bar{\eta}; \ FC_1^{-1}: \eta = \frac{u\bar{v} - \bar{u}v + \mathrm{i}(\bar{u} - u)}{\bar{v} - v}, \tag{14}$$

where ω'_1 is the Kähler two-form (9), $G^J_1(\mathbb{R})$-invariant to the action (10), and

$$\omega'_0 = \omega_{\mathcal{X}_1} + \omega_{\mathbb{C}}, \ \mathrm{i}d\omega_{\mathcal{X}_1} = \frac{2k}{(v - \bar{v})^2} dv \wedge d\bar{v}. \tag{15}$$

Proof. We use the transformation (8) and the (Eichler-Zagier) coordinates [5] $v = x + iy; \ u = pv + q, \ x, p, q, y \in \mathbb{R}, y > 0$, and come back from v to w. Let $z = 2\mathrm{i}(\mathrm{i}+v)^{-1}(pv+q)$, where $v = -\mathrm{i}(w-1)^{-1}(w+1)$. We have $z = q + \mathrm{i}p + w(-q + \mathrm{i}p)$, and if denote $\eta = q + \mathrm{i}p$, where $q, p \in \mathbb{R}$, then $z = \eta - w\bar{\eta}$, with the same η as in (6), and $A = d\eta - wd\bar{\eta}$. The last term in (6) becomes $(1 - |w|^2)^{-1} A \wedge \bar{A} = d\eta \wedge d\bar{\eta} = 2\mathrm{i}dp \wedge dq$. Vice-versa, we have $d\eta = (1 - |w|^2)^{-1}(A + w\bar{A})$, with A given in (6).

For the second assertion, we introduce the transformation (8) $z = 2\mathrm{i}u(v+\mathrm{i})^{-1}$ in (11) and we get: $2\mathrm{i}(u-\bar{u}) = (\eta-\bar{\eta})(v-\bar{v})$. Than G in (9) becomes $G = \frac{1}{2\mathrm{i}}[(v+\mathrm{i})d\eta - (v-\mathrm{i})d\bar{\eta}]$ and we get (15). □

4. Motion on the Siegel-Jacobi manifolds $\mathcal{D}_1^J$ and $\mathcal{X}_1^J$

We consider a homogeneous manifold $M = G/H$ endowed with a G-invariant Kähler two-form $\omega(z)$ deduced form the scalar product of coherent state vectors $e_z \in \mathfrak{H}$, obtained from a unitary irreducible representation of G on the Hilbert space $\mathfrak{H}$ [6]. Passing on from the dynamical system problem in the Hilbert space $\mathfrak{H}$ to the corresponding one on M (*dequantization*), the dynamical system on M is a classical one. The motion on the classical phase space M can be described by the Hamiltonian equations of motion $\dot{z}_\alpha = \mathrm{i}\{\mathcal{H}, z_\alpha\},\ \alpha \in \Delta_+$, where $\mathcal{H} = \langle e_z, e_z\rangle^{-1}\langle e_z|\boldsymbol{H}|\mathrm{e}_z\rangle$ is the classical Hamiltonian (the covariant symbol) attached to the quantum Hamiltonian $\boldsymbol{H}$ [26],[27]. We consider an algebraic Hamiltonian linear in the generators $\boldsymbol{X}_\lambda$ of the group of symmetry G, $\boldsymbol{H} = \sum_{\lambda\in\Delta}\epsilon_\lambda\boldsymbol{X}_\lambda$. Let us suppose that $\boldsymbol{X}_\lambda$ can be expressed in a local system of coordinates as a holomorphic first-order differential operator with polynomial coefficients,

$$\mathbb{X}_\lambda = P_\lambda + \sum_{\beta\in\Delta_+} Q_{\lambda,\beta}\frac{\partial}{\partial z_\beta}, \lambda \in \Delta. \tag{16}$$

We recall that [29]

Proposition 4. *On the homogeneous manifold $M = G/H$ on which the holomorphic representation* (16) *is true, the classical motion and the quantum evolution generated by the linear Hamiltonian $\boldsymbol{H}$ are described by the same equation of motion* (17)

$$\mathrm{i}\dot{z}_\alpha = \sum_{\lambda\in\Delta}\epsilon_\lambda Q_{\lambda,\alpha},\ \alpha \in \Delta_+, \tag{17}$$

We consider a linear hermitian Hamiltonian in the generators of the group G_1^J

$$\boldsymbol{H} = \epsilon_a\boldsymbol{a} + \bar{\epsilon}_a\boldsymbol{a}^\dagger + \epsilon_0\boldsymbol{K}_0 + \epsilon_+\boldsymbol{K}_+ + \epsilon_-\boldsymbol{K}_-,\ \bar{\epsilon}_+ = \epsilon_{:}\ \epsilon_0 = \bar{\epsilon_0}. \tag{18}$$

The general scheme [28, 29] associates to elements of the Lie algebra $\mathfrak{g}$ first-order holomorphic differential operators with polynomial coefficients, $X \in \mathfrak{g} \to \mathbb{X}$, and for G_1^J we have [8]:

Lemma 5. *The differential action of the generators of the Jacobi algebra $\mathfrak{g}_1^J$ is given by the formulas:*

$$\boldsymbol{a} = \frac{\partial}{\partial z}; \boldsymbol{a}^+ = z + w\frac{\partial}{\partial z},\ z, w \in \mathbb{C}, |w| < 1;$$
$$\mathbb{K}_- = \frac{\partial}{\partial w}, \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}.$$

With Lemma 5, Proposition 4, and Proposition 3, we get

Proposition 6. *The linear hermitian Hamiltonian* (18) *generates:*

a) *the differential equations of motion on the Siegel-Jacobi disk* $\mathcal{D}^J_1$*:*

$$\begin{aligned} i\dot{z} &= \epsilon_a + \bar{\epsilon}_a w + (\tfrac{\epsilon_0}{2} + \epsilon_+ w)z, \\ i\dot{w} &= \epsilon_- + \epsilon_0 w + \epsilon_+ w^2; \end{aligned} \tag{19}$$

b) *the equations of motion in* (v,u) *on the manifold* $\mathcal{X}^J_1$*, obtained from the equations* (19) *by the partial Cayley transform* (8)*:*

$$\begin{aligned} 2\dot{v} &= (\epsilon_0+\epsilon_++\epsilon_-)v^2 + 2\mathrm{i}(\epsilon_- - \epsilon_+)v + \epsilon_0 - \epsilon_- - \epsilon_+, \\ 2\dot{u} &= (\epsilon_a+\bar{\epsilon}_a)v + \mathrm{i}(\epsilon_a - \bar{\epsilon}_a) + [(\epsilon_0+\epsilon_++\epsilon_-)v + \mathrm{i}(\epsilon_- - \epsilon_+)]u; \end{aligned} \tag{20}$$

c) *the decoupled equations of motion in* $(\eta, w) \in \mathbb{C} \times \mathcal{D}_1$*:*

$$\begin{aligned} i\dot{\eta} &= \epsilon_a + \epsilon_- \bar{\eta} + \tfrac{\epsilon_0}{2}\eta, \\ i\dot{w} &= \epsilon_- + \epsilon_0 w + \epsilon_+ w^2; \end{aligned} \tag{21}$$

d) *and the decoupled differential equations in* $(\eta, w) \in \mathbb{C} \times \mathcal{X}_1$*:*

$$\begin{aligned} i\dot{\eta} &= \epsilon_a + \epsilon_- \bar{\eta} + \tfrac{\epsilon_0}{2}\eta, \ \ \eta \in \mathbb{C}, \\ -2\dot{v} &= (\epsilon_0 + \epsilon_+ + \epsilon_-)v^2 + 2\mathrm{i}(\epsilon_- - \epsilon_+)v + \epsilon_0 - \epsilon_- - \epsilon_+. \end{aligned} \tag{22}$$

For constant coefficients, the Riccati equation in (19) *has the solution*

$$w(t) = \frac{w_1 C_1 \mathrm{e}^{\frac{\mathrm{i}\sqrt{\Delta}}{2}t} + w_2 C_2 \mathrm{e}^{-\frac{\mathrm{i}\sqrt{\Delta}}{2}t}}{\epsilon_+(C_1 \mathrm{e}^{\frac{\mathrm{i}\sqrt{\Delta}}{2}t} + C_2 \mathrm{e}^{-\frac{\mathrm{i}\sqrt{\Delta}}{2}t})}, w_{1,2} = \frac{-\epsilon_0 \pm \sqrt{\Delta}}{2}, \Delta = \epsilon_0^2 - 4\epsilon_+\epsilon_-, \tag{23}$$

and the condition $w(t) \in \mathcal{D}_1$ *imposes the restrictions:*

$$|\frac{C_1}{C_2}| > \sqrt{\frac{w_2}{w_1}} = \frac{1+\sqrt{1-\delta}}{\sqrt{\delta}}, \ \epsilon_0 > 0, \ \Delta > 0, \ \delta = 4\frac{\epsilon_+\epsilon_-}{\epsilon_0^2} < 1. \tag{24}$$

The solution $\eta(t)$ *of the first differential equation* (21) *is:*

$$\eta(t) = M\mathrm{e}^{\mathrm{i}\frac{\sqrt{\Delta}}{2}t} + N\mathrm{e}^{-\mathrm{i}\frac{\sqrt{\Delta}}{2}t} + P, \quad \text{where} \tag{25a}$$

$$M = -\mathrm{i}\frac{q\alpha}{r\sqrt{\Delta}}(\epsilon_- + w_1); \ \ N = \mathrm{i}\frac{q\beta}{r\sqrt{\Delta}}(\epsilon_- + w_2), \tag{25b}$$

$$\frac{\alpha}{\bar{\beta}} = \frac{\epsilon_-(\epsilon_+ + w_2)}{w_2(\epsilon_- + w_1)} = \frac{w_1(\epsilon_+ + w_2)}{\epsilon_+(\epsilon_- + w_1)}, \tag{25c}$$

$$P = \frac{4\epsilon_-\bar{\epsilon}_a - 2\epsilon_0\epsilon_a}{\Delta}, \ \ r = \frac{1}{2}(\epsilon_- + \epsilon_+ - \epsilon_o), \tag{25d}$$

$$q = -\frac{\epsilon_0}{4}(\epsilon_a + \bar{\epsilon}_a) + \frac{1}{2}(\epsilon_a\epsilon_+ + \bar{\epsilon}_a\epsilon_-). \tag{25e}$$

The solution z *of the first differential equation* (19) *is given by* $z(t) = \eta(t) - w(t)\bar{\eta}(t)$*, where* $\eta(t)$ *has the expression given by* (25)*, while the solution* $w(t)$ *of second equation* (19) *is given by* (23)*.*

The solution v *of the second equation* (22) *is obtained from the solution* $w(t)$ (23) *of second equation* (19) *via the Cayley transform* $v = \mathrm{i}(1-w)^{-1}(1+w)$*.*

The second equation (19) in w (the first equation in v in (20)) is a *Riccati equation* on $\mathcal{D}_1$ (respectively, on $\mathcal{X}_1$). Remark that the dynamics on the Siegel disk $\mathcal{D}_1$, determined by the Hamiltonian (18), *linear in the generators of the Jacobi group G^J_1, depends* **only** *on the generators of the group* SU(1, 1). The Riccati equation on $\mathcal{D}_1$ in w appears in literature, see, e.g., equation (18.2.8) in [6] in the context of quantum oscillator with variable frequency.

5. The fundamental conjecture for the Siegel-Jacobi manifolds

We consider the Lie algebra $\mathfrak{g}^J_n := \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ of the Jacobi group G^J_n, generated by $a^{\dagger}_i, a_i, K^{0,+,-}_{ij}$, $i, j = 1, \dots, n$, and the coherent state vectors [7]

$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} = \sum_i z_i\boldsymbol{a}^{\dagger}_i + \sum_{ij} w_{ij}\boldsymbol{K}^+_{ij},\ z \in \mathbb{C}^n,\ W \in \mathcal{D}_n,$$

defined on the Siegel-Jacobi ball $\mathcal{D}^J_n$. The Kähler two-form on $\mathcal{D}^J_n$

$$\begin{aligned} -\mathrm{i}\omega_n &= \tfrac{k}{2}\mathrm{Tr}(F \wedge \bar{F}) + \mathrm{Tr}(A^t\bar{M} \wedge \bar{A}), A = \mathrm{d}\, z + \mathrm{d}\, W\bar{\eta},\\ F &= M\,\mathrm{d}\, W, M = (1 - W\bar{W})^{-1},\ \eta = M(z + W\bar{z}), \end{aligned} \tag{26}$$

is G^J_n-invariant under the action $(g, \alpha)\cdot(z, W) \to (z_1, W_1) \in \mathbb{C}^n \times \mathcal{D}_n$, where:

$$\begin{aligned} z_1 &= (Wb^* + a^*)^{-1}(z + \alpha - W\bar{\alpha}),\ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}},\\ W_1 &= (aW + b)(\bar{b}W + \bar{a})^{-1} = (Wb^* + a^*)^{-1}(b^t + Wa^t). \end{aligned} \tag{27}$$

If A is matrix, then A^t denotes its transpose, and $A^* = \bar{A}^t$.

We consider also the real Jacobi group $G^J_n(\mathbb{R}) = \mathrm{Sp}(n, \mathbb{R}) \ltimes H_n$ and the Siegel-Jacobi space $\mathcal{X}^J_n := \mathcal{X}_n \times \mathbb{R}^{2n}$ [20], where $\mathcal{X}_n$ is the Siegel upper half-plane realized as

$$\mathcal{X}_n := \{v \in M(n, \mathbb{C}) | v = s + \mathrm{i}r, s, r \in M(n, \mathbb{R}), r > 0, s^t = s; r^t = r\}.$$

Let $g = (V, l) \in G^J_n(\mathbb{R})_0$, i.e., $V \in \mathrm{Sp}(n, \mathbb{R})$, $l = (l_1, l_2) \in \mathbb{R}^{2n}$, and $v \in \mathcal{H}_n$, $u \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$. *The action of the group $G^J_n(\mathbb{R})_0$* on $\mathcal{X}^J_n$, $(V, l)\cdot(v, u) \to (v_1, u_1)$, is given by the formulae:

$$v_1 = (Av+B)(Cv+D)^{-1}, u_1 = (vC^t+D^t)^{-1}(u+vl_1+l_2), V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{28}$$

Under the partial Cayley transform $\mathcal{X}^J_n \to \mathcal{D}^J_n$,

$$W = (v - i)(v + i)^{-1};\ z = (v + i)^{-1}2iu, \tag{29}$$

the Kähler two-form ω_n on $\mathcal{D}^J_n$ (26) *becomes the $G^J_n(\mathbb{R})_0$-invariant Kähler two-form on $\mathcal{X}^J_n$,*

$$\begin{aligned} -\mathrm{i}\omega'_n &= \frac{k}{2}\mathrm{Tr}(H \wedge \bar{H}) + \frac{2}{\mathrm{i}}\mathrm{Tr}(G^tD \wedge \bar{G}), \quad \text{where}\\ D &= (\bar{v} - v)^{-1},\ H = Ddv;\ G = du - dvD(\bar{u} - u). \end{aligned} \tag{30}$$

In (29), we make the change of variables $u = vp + q$, $p, q \in \mathbb{R}^n$ and $v = -\mathrm{i}(W + 1)^{-1}(W + 1)$. Then

$$z = \eta - W\bar{\eta}, \tag{31}$$

where $\eta = q + \mathrm{i}p \in \mathbb{C}^n$, and A in (26) is $A = \mathrm{d}\,\eta - W\,\mathrm{d}\,\bar{\eta}$. We get for G^J_n the analogous of Proposition 3:

Proposition 7. *Under the homogeneous Kähler transform FC* (31)*, the Kähler two-form* (26) *on* $\mathcal{D}^J_n$*,* G^J_n*-invariant to the action* (27)*, becomes the Kähler two-form on* $\mathcal{D}_n \times \mathbb{C}^n$

$$-\mathrm{i}\omega_{n,0} = \tfrac{k}{2}\mathrm{Tr}(F \wedge \bar{F}) + \mathrm{Tr}(\mathrm{d}\,\eta^t \wedge \mathrm{d}\,\bar{\eta}), \tag{32}$$

invariant to the G^J_n*-action on* $\mathcal{D}_n \times \mathbb{C}^n$*,* $(g, \alpha) \cdot (\eta, W) \to (\eta_1, W_1)$*, with* W_1 *given in* (27) *and*

$$\eta_1 = a(\eta + \alpha) + b(\bar{\eta} + \bar{\alpha}). \tag{33}$$

Under the homogeneous Kähler transform

$$FC_1^{-1} : \eta = (\bar{v} - \mathrm{i})(\bar{v} - v)^{-1}(v - \mathrm{i})[(v - \mathrm{i})^{-1}u - (\bar{v} - \mathrm{i})^{-1}\bar{u}]. \tag{34}$$

the Kähler two-form (30) *becomes a Kähler two-form on* $\mathcal{X}_n \times \mathbb{C}^n$

$$-\mathrm{i}\omega'_{n,0} = \tfrac{k}{2}\mathrm{Tr}(H \wedge \bar{H}) + \mathrm{Tr}(\mathrm{d}\,\eta^t \wedge \mathrm{d}\,\bar{\eta}),\ H = (\bar{v} - v)^{-1}\,\mathrm{d}\,v. \tag{35}$$

Now we consider a hermitian Hamiltonian linear in the generators of the group G^J_n

$$\begin{aligned} \boldsymbol{H} &= \epsilon_i \boldsymbol{a}_i + \bar{\epsilon}_i \boldsymbol{a}^+_i + \epsilon^0_{ij}\boldsymbol{K}^0_{ij} + \epsilon^-_{ij}\boldsymbol{K}^-_{ij} + \epsilon^+_{ij}\boldsymbol{K}^+_{ij}, \quad \text{where} \\ (\epsilon^0)^\dagger &= \epsilon^0;\ \epsilon^- = (\epsilon^-)^t;\ \epsilon^+ = (\epsilon^+)^t;\ (\epsilon^+)^\dagger = \epsilon^-. \end{aligned} \tag{36}$$

With Proposition 4, the differential realization of the generators of G^J_n [7], and Proposition 7, we get

Proposition 8. *The linear hermitian Hamiltonian* (36) *generates:*

a) *the matrix equations of motion on* $\mathcal{D}^J_n$*:*

$$\begin{aligned} \mathrm{i}\dot{z} &= \epsilon + W\bar{\epsilon} + \frac{1}{2}(\epsilon^0)^t z + W\epsilon^+ z,\ z \in \mathbb{C}^n, \\ \mathrm{i}\dot{W} &= \epsilon^- + \frac{1}{2}[W\epsilon^0 + (\epsilon^0)^t W] + W\epsilon^+ W,\ W \in \mathcal{D}_n; \end{aligned} \tag{37}$$

b) *the coupled matrix equations in* (u, v) *on* $\mathcal{X}^J_n = \mathbb{C}^n \times \mathcal{X}_n$*:*

$$\begin{aligned} -2\dot{u} &= v(\epsilon + \bar{\epsilon}) + \mathrm{i}(\epsilon - \bar{\epsilon}) + \left[v\left(\frac{\epsilon^0 + (\epsilon^0)^t}{2} + \epsilon^+ + \epsilon^-\right) + \mathrm{i}\left(\frac{(\epsilon^0)^t - \epsilon^0}{2} + \epsilon^- - \epsilon^+\right)\right]u, \\ -2\dot{v} &= \frac{\epsilon^0 + (\epsilon^0)^t}{2} - (\epsilon^- + \epsilon^+) + \mathrm{i}v\left(\frac{\epsilon^0 - (\epsilon^0)^t}{2} + \epsilon^- - \epsilon^+\right) \\ &\quad + \mathrm{i}\left(\frac{-\epsilon^0 + (\epsilon^0)^t}{2} + \epsilon^- - \epsilon^+\right)v + v\left(\frac{\epsilon^0 + (\epsilon^0)^t}{2} + \epsilon^- + \epsilon^+\right)v; \end{aligned} \tag{38}$$

c) *the decoupled equations in* $(\eta, W) \in \mathbb{C}^n \times \mathcal{D}_n$:

$$\begin{aligned} \mathrm{i}\dot{\eta} &= \epsilon + \epsilon^- \bar{\eta} + \tfrac{1}{2}(\epsilon^0)^t \eta, \\ \mathrm{i}\dot{W} &= \epsilon^- + \tfrac{1}{2}[W\epsilon^0 + (\epsilon^0)^t W] + W\epsilon^+ W; \end{aligned} \tag{39}$$

d) *and the decoupled matrix equations in* $(\eta, v) \in \mathbb{C}^n \times \mathcal{X}_n$:

$$\begin{aligned} \mathrm{i}\dot{\eta} &= \epsilon + \epsilon^- \bar{\eta} + \frac{1}{2}(\epsilon^0)^t \eta, \\ -2\dot{v} &= \frac{\epsilon^0 + (\epsilon^0)^t}{2} - (\epsilon^- + \epsilon^+) + \mathrm{i}v\left(\frac{\epsilon^0 - (\epsilon^0)^t}{2} + \epsilon^- - \epsilon^+\right) \\ &\quad + \mathrm{i}\left(\frac{-\epsilon^0 + (\epsilon^0)^t}{2} + \epsilon^- - \epsilon^+\right)v + v\left(\frac{\epsilon^0 + (\epsilon^0)^t}{2} + \epsilon^- + \epsilon^+\right)v. \end{aligned} \tag{40}$$

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Exhausting Formal Quantization Procedures

Vasily A. Dolgushev

Abstract. In paper [1] the author introduced stable formality quasi-isomorphisms and described the set of its homotopy classes. This result can be interpreted as a complete description of formal quantization procedures. In this note we give a brief exposition of stable formality quasi-isomorphisms and prove that every homotopy class of stable formality quasi-isomorphisms contains a representative which admits globalization. This note is loosely based on the talk given by the author at XXX Workshop on Geometric Methods in Physics in Białowieża, Poland.

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1. Introduction

In seminal paper [2] M. Kontsevich constructed an L_∞ quasi-isomorphism from the graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^d$ to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ for the polynomial algebra $A = \mathbb{R}[x^1, x^2, \dots, x^d]$. This result implies that equivalence classes of star-products on $\mathbb{R}^d$ are in bijection with the equivalence classes of formal Poisson structures on $\mathbb{R}^d$. This theorem also implies that Hochschild cohomology of a deformation quantization algebra is isomorphic to the Poisson cohomology of the corresponding formal Poisson structure.

In view of these consequences, we will think about L_∞ quasi-isomorphisms from the graded Lie algebra of polyvector fields on the affine space $\mathbb{R}^d$ to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ as *formal quantization procedures.*

Following [3] one can define a natural notion of homotopy equivalence on the set of L_∞-morphisms between dg Lie algebras (or even L_∞-algebras). Furthermore, according to Lemma B.5 from [4], homotopy equivalent L_∞ quasi-morphisms for $C^\bullet(A)$ give the same bijection between the set of equivalence classes of star-products and the set of equivalence classes of formal Poisson structures. Thus, for the purposes of applications, we should only be interested in homotopy classes of formality quasi-isomorphisms.

In paper [1] the author developed a framework of what he calls *stable formality quasi-isomorphisms* (*SFQ*) and showed that homotopy classes of such SFQ's form a torsor for the group which is obtained by exponentiating the Lie algebra $H^0(\mathsf{GC})$ where GC is the graph complex introduced by M. Kontsevich in [5, Section 5]. Any SFQ gives us an L_∞ quasi-isomorphism for the Hochschild cochains of $A = \mathbb{R}[x^1, x^2, \dots, x^d]$ in all[1] dimensions d simultaneously. Moreover, homotopy equivalent SFQ's give homotopy equivalent L_∞ quasi-isomorphisms for the Hochschild cochains of $A = \mathbb{R}[x^1, x^2, \dots, x^d]$. Thus the main result (Theorem 6.2) of [1] can be interpreted as a complete description of formal quantization procedures in the stable setting.

In the next section we remind the full (directed) graph complex and its relation to Kontsevich's graph complex GC [5, Section 5]. In Section 3 we give a brief exposition of stable formality quasi-isomorphisms (SFQ). Finally, in Section 4 we prove that every SFQ is homotopy equivalent to an SFQ which admits globalization.

Notation and conventions. In this note we assume that the ground field $\mathbb{K}$ contains the field of reals. For most of algebraic structures considered in this note, the underlying symmetric monoidal category is the category of unbounded cochain complexes of $\mathbb{K}$-vector spaces. For a cochain complex $\mathcal{V}$ we denote by $\mathbf{s}\mathcal{V}$ (resp. by $\mathbf{s}^{-1}\mathcal{V}$) the suspension (resp. the desuspension) of $\mathcal{V}$. In other words,

$$\big(\mathbf{s}\mathcal{V}\big)^\bullet = \mathcal{V}^{\bullet-1}, \qquad \big(\mathbf{s}^{-1}\mathcal{V}\big)^\bullet = \mathcal{V}^{\bullet+1}.$$

$C^\bullet(A)$ denotes the Hochschild cochain complex of an associative algebra (or more generally an A_∞-algebra) A with coefficients in A. For a commutative ring R and an R-module V we denote by $S_R(V)$ the symmetric algebra of V over R.

Given an operad $\mathcal{O}$, we denote by $\circ_i$ the elementary operadic insertions:

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(k) \to \mathcal{O}(n+k-1), \qquad 1 \le i \le n.$$

The notation $\mathrm{Sh}_{p,q}$ is reserved for the set of (p,q)-shuffles in S_{p+q}. A graph is *directed* if each edge carries a chosen direction. A graph Γ with n vertices is called *labeled* if Γ is equipped with a bijection between the set of its vertices and the set $\{1, 2, \dots, n\}$. ε denotes a formal deformation parameter.

2. The full directed graph complex dfGC

In this section we recall from [6] an extended version dfGC of Kontsevich's graph complex GC [5, Section 5]. For this purpose, we first introduce a collection of auxiliary sets $\{\mathrm{dgra}(n)\}_{n\ge1}$. An element of dgra_n is a directed labeled graph Γ with n vertices and with the additional piece of data: the set of edges of Γ is equipped with a total order. An example of an element in dgra_4 is shown in Figure 1.

Next, we introduce a collection of graded vector spaces $\{\mathsf{dGra}(n)\}_{n\ge1}$. The space $\mathsf{dGra}(n)$ is spanned by elements of dgra_n, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$

[1] In fact they are also defined for any $\mathbb{Z}$-graded affine space.

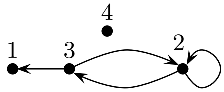


FIGURE 1. The edges are equipped with the order
$(3,1) < (3,2) < (2,3) < (2,2)$.

where the graphs Γ^σ and Γ correspond to the same directed labeled graph but differ only by permutation σ of edges. We also declare that the degree of a graph Γ in $\mathsf{dGra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in Γ. For example, the graph Γ on figure 1 has 4 edges. Thus its degree is -4.

Following [6], the collection $\{\mathsf{dGra}(n)\}_{n\geq 1}$ forms an operad. The symmetric group S_n acts on $\mathsf{dGra}(n)$ in the obvious way by rearranging labels and the operadic multiplications are defined in terms of natural operations of erasing vertices and attaching edges to vertices.

The operad dGra can be upgraded to a 2-colored operad KGra whose spaces[2] are formal linear combinations of graphs used by M. Kontsevich in [2].

We define the graded vector space dfGC by setting

$$\mathsf{dfGC} = \prod_{n\geq 1} \mathbf{s}^{2n-2}\Big(\mathsf{dGra}(n)\Big)^{S_n}. \tag{1}$$

Next, we observe that the formula

$$\Gamma \bullet \widetilde{\Gamma} = \sum_{\sigma\in \mathrm{Sh}_{k,n-1}} \sigma\big(\Gamma \circ_1 \widetilde{\Gamma}\big) \tag{2}$$

$$\Gamma \in \Big(\mathsf{dGra}(n)\Big)^{S_n}, \qquad \widetilde{\Gamma} \in \Big(\mathsf{dGra}(k)\Big)^{S_k}$$

defines a degree zero $\mathbb{K}$-bilinear operation on $\bigoplus_{n\geq 1} \mathbf{s}^{2n-2}\Big(\mathsf{dGra}(n)\Big)^{S_n}$ which extends in the obvious way to the graded vector space dfGC (1).

It is not hard to show that the operation (2) satisfies axioms of the pre-Lie algebra and hence dfGC is naturally a Lie algebra with the bracket give by the formula

$$[\gamma, \widetilde{\gamma}] = \gamma \bullet \widetilde{\gamma} - (-1)^{|\gamma||\widetilde{\gamma}|}\, \widetilde{\gamma} \bullet \gamma\,, \tag{3}$$

where γ and $\widetilde{\gamma}$ are homogeneous vectors in dfGC.

A direct computation shows that the degree 1 vector

$\Gamma_{\bullet\!-\!\bullet} =$  (4)

satisfies the Maurer-Cartan equation $[\,\Gamma_{\bullet\!-\!\bullet}, \Gamma_{\bullet\!-\!\bullet}\,] = 0$.

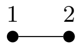
[2]For more details, we refer the reader to [1, Section 3].

Thus, dfGC forms a dg Lie algebra with the bracket (3) and the differential

$$\partial = [\,\Gamma_{\bullet\!-\!\bullet},\ \,]\,. \tag{5}$$

Definition 1. The cochain complex $(\mathsf{dfGC}, \partial)$ is called the full directed graph complex.

Let us observe that every undirected labeled graph Γ with n vertices and with a chosen order on the set of its edges can be interpreted as the sum of all directed labeled graphs Γ_α in $\mathrm{dgra}(n)$ from which the graph Γ is obtained by forgetting directions on edges. For example,

$\Gamma_{\bullet\!-\!\bullet} =$  (6)

Thus, using undirected labeled graphs we may form a suboperad Gra inside dGra and the sub- dg Lie algebra

$$\mathsf{fGC} = \prod_{n \ge 1} \mathbf{s}^{2n-2}\Big(\mathsf{Gra}(n)\Big)^{S_n} \quad \subset \quad \mathsf{dfGC} \tag{7}$$

Definition 2 (M. Kontsevich, [5]). *Kontsevich's graph complex* GC is the subcomplex

$$\mathsf{GC} \subset \mathsf{fGC} \tag{8}$$

formed by (possibly infinite) linear combinations of connected graphs Γ satisfying these two properties: *each vertex of* Γ *has valency* ≥ 3*, and the complement to any vertex is connected.*

It is easy to see that GC is a sub- dg Lie algebra of fGC. Furthermore, following[3] [6] we have

Theorem 1 (T. Willwacher, [6]). *The cohomology of* dfGC *can be expressed in terms of cohomology of* GC*. More precisely,*

$$H^\bullet(\mathsf{dfGC}) = \mathbf{s}^{-2}\, S\big(\mathbf{s}^2\,\mathcal{H}\big) \tag{9}$$

where

$$\mathcal{H} = H^\bullet(\mathsf{GC}) \ \oplus\ \bigoplus_{m \ge 0} \mathbf{s}^{4m-1}\mathbb{K}\,.$$

Using decomposition (9), it is not hard to see that

$$H^0(\mathsf{dfGC}) \cong H^0(\mathsf{GC}) \tag{10}$$

and the Lie algebra $H^0(\mathsf{dfGC})$ is pro-nilpotent.

[3] See lecture notes [7] for more detailed exposition.

3. Stable formality quasi-isomorphisms

Let $A = \mathbb{K}[x^1, x^2, \dots, x^d]$ be the algebra of functions on the affine space $\mathbb{K}^d$ and let $V_A^\bullet$ be the algebra of polyvector fields on $\mathbb{K}^d$

$$V_A^\bullet = S_A\big(\mathbf{s}\,\mathrm{Der}(A)\big)\,. \tag{11}$$

Recall that $V_A^\bullet = \mathbb{K}[x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d]$ is a free commutative algebra over $\mathbb{K}$ in d generators $x^1, x^2, \dots, x^d$ of degree zero and d generators $\theta_1, \theta_2, \dots, \theta_d$ of degree one.

It is know that $V_A^{\bullet+1}$ is a graded Lie algebra. The Lie bracket on $V_A^{\bullet+1}$ is given by the formula:

$$[v, w]_S = (-1)^{|v|} \sum_{i=1}^{d} \frac{\partial v}{\partial \theta_i} \frac{\partial w}{\partial x^i} - (-1)^{|v||w|+|w|} \sum_{i=1}^{d} \frac{\partial w}{\partial \theta_i} \frac{\partial v}{\partial x^i}\,. \tag{12}$$

It is called the *Schouten bracket.*

In plain English an L_∞-morphism U from $V_A^{\bullet+1}$ to $C^{\bullet+1}(A)$ is an infinite collection of maps

$$U_n : \big(V_A^{\bullet+1}\big)^{\otimes\, n} \to C^{\bullet+1}(A)\,, \qquad n \ge 1 \tag{13}$$

compatible with the action of symmetric groups and satisfying an intricate sequence of quadratic relations. The first relation says that U_1 is a map of cochain complexes, the second relation says that U_1 is compatible with the Lie brackets up to homotopy with U_2 serving as a chain homotopy and so on.

Kontsevich's construction of such a sequence (13) is "natural" in the following sense: given polyvector fields $v_1, v_2, \dots, v_n \in V_A^{\bullet+1}$, the value

$$U_n\big(v_1, v_2, \dots, v_n\big)(a_1, a_2, \dots, a_k) \tag{14}$$

of the cochain $U_n(v_1, v_2, \dots, v_n)$ on polynomials $a_1, a_2, \dots, a_k \in A$ is obtained via contracting all indices of derivatives of various orders of $v_1, \dots, v_n, a_1, \dots, a_k$ in such a way that the resulting map

$$(V_A^\bullet)^{\otimes\, n} \otimes A^{\otimes\, k} \to A$$

is $\mathfrak{gl}_d(\mathbb{K})$-equivariant. Thus each term in U_n can be encoded by a directed graph with two types of vertices: vertices of one type are reserved for polyvector fields and vertices of another type are reserved for polynomials.

Motivated by this observation, the author introduced in [1] a notion of *stable formality quasi-isomorphism* (*SFQ*) which formalizes L_∞ quasi-isomorphisms U for Hochschild cochains satisfying this property: *each term in U_n is encoded by a graph with two types of vertices and all the desired relations on U_n's hold universally, i.e., on the level of linear combinations of graphs.*

The precise definition of SFQ is given in terms of 2-colored dg operads OC and KGra . The later operad KGra is a 2-colored extension of the operad dGra which is "assembled" from graphs used by M. Kontsevich in [2]. This operad comes with a natural action on the pair $(V_A^{\bullet+1}, A = \mathbb{K}[x^1, \dots, x^d])$. The operad OC governs open-closed homotopy algebras introduced in [8] by H. Kajiura and J. Stasheff. We recall that an open-closed homotopy algebra is a pair $(\mathcal{V}, \mathcal{A})$ of cochain complexes equipped with the following data:

- An L_∞-structure on $\mathcal{V}$;
- an A_∞-structure on $\mathcal{A}$; and
- an L_∞-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of the A_∞-algebra $\mathcal{A}$.

Since the operad KGra acts on the pair $(V_A^{\bullet+1}, A = \mathbb{K}[x^1, \dots, x^d])$, any morphism of dg operads

$$F : \mathsf{OC} \to \mathsf{KGra} \tag{15}$$

gives us an L_∞-structure on $V_A^{\bullet+1}$, an A_∞-structure on A and an L_∞ morphism from $V_A^{\bullet+1}$ to $C^\bullet(A)$.

An SFQ is defined as a morphism (15) of dg operads satisfying three boundary conditions. The first condition guarantees that the L_∞-algebra structure on $V_A^{\bullet+1}$ induced by F coincides with the Lie algebra structure given by the Schouten bracket (12). The second condition implies that the A_∞-algebra structure on A coincides with the usual associative (and commutative) algebra structure on polynomials. Finally, the third condition ensures that the L_∞-morphism

$$U : V_A^{\bullet+1} \rightsquigarrow C^{\bullet+1}(A)$$

induced by F starts with the Hochschild-Kostant-Rosenberg embedding. In particular, the last condition implies that U is an L_∞ quasi-isomorphism.

Kontsevich's construction [2] provides us with an example of an SFQ over any extension of the field of reals.[4]

In paper [1] the author also defined the notion of homotopy equivalence for SFQ's. This notion is motivated by the property that L_∞ quasi-isomorphisms

$$U, \widetilde{U} : V_A^{\bullet+1} \rightsquigarrow C^{\bullet+1}(A)$$

corresponding to homotopy equivalent SFQ's F and $\widetilde{F}$ are connected by a homotopy which "admits a graphical expansion" in the above sense.

Following [5] we have a chain map Θ from the full (directed) graph complex dfGC to the deformation complex of the dg Lie algebra $V_A^{\bullet+1}$ of polyvector fields. In particular, every degree zero cocycle in dfGC produces an L_∞-derivation of $V_A^{\bullet+1}$. Exponentiating these L_∞-derivations we get an action of the (pro-unipotent) group

$$\exp\left(\mathsf{dfGC}^0 \cap \ker \partial\right)$$

on the set of L_∞ quasi-isomorphisms

$$U : V_A^{\bullet+1} \rightsquigarrow C^{\bullet+1}(A) \tag{16}$$

for $A = \mathbb{K}[x^1, \dots, x^d]$. Namely, given a cocycle $\gamma \in \mathsf{dfGC}^0$, the action of $\exp(\gamma)$ is defined by the formula

$$U \mapsto U \circ \exp\left(-\Theta(\gamma)\right), \tag{17}$$

where Θ is the chain map from dfGC to the deformation complex of $V_A^{\bullet+1}$.

[4]The existence of an SFQ over rationals is proved in papers [9] and [10].

In [1], it was proved that the action (17) descends to an action of the (pro-unipotent) group

$$\exp\left(H^0(\mathsf{dfGC})\right) \tag{18}$$

on the set of homotopy classes of SFQ's. Moreover,

Theorem 2 (Theorem 6.2, [1]). *The group* (18) *acts simply transitively on the set of homotopy classes of SFQ's.*

In the view of philosophy outlined in the Introduction, this result can be interpreted as a complete description of formal quantization procedures.

Remark 3. According to a recent result [6, Thm. 1] of T. Willwacher, $\exp\left(H^0(\mathsf{GC})\right)$ is isomorphic to the Grothendieck-Teichmueller group GRT introduced by V. Drinfeld in [11]. Thus, combining this result with Theorem 2, we conclude that formal quantization procedures are "governed" by the group GRT.

Remark 4. In recent preprint [12] Thomas Willwacher computes stable cohomology of the graded Lie algebra of polyvector fields with coefficients in the adjoint representation. His computations partially justify the name "stable formality quasi-isomorphism" chosen by the author in [1]. In particular, Thomas Willwacher mentions in [12] a possibility to deduce the part about transitivity from Theorem 2 in a more conceptual way.

4. Globalization of stable formality quasi-isomorphisms

Given an L_∞ quasi-isomorphism (16) for $A = \mathbb{K}[x^1, \dots, x^d]$ we can ask the question of whether we can use it to construct a sequence of L_∞ quasi-isomorphisms which connects the sheaf $V_X^{\bullet+1}$ of polyvector fields to the sheaf $\mathcal{D}_X^{\bullet+1}$ of polydifferential operators on a smooth algebraic variety X over $\mathbb{K}$. There are several similar constructions [13], [14], [15] which allow us to produce such a sequence under the assumption that the L_∞ quasi-isomorphism (16) satisfies the following properties:

- **A)** One can replace $A = \mathbb{K}[x^1, \dots, x^d]$ in (16) by its completion $A_{\mathrm{formal}} = \mathbb{K}[[x^1, \dots, x^d]]$;
- **B)** the structure maps U_n of U are $\mathfrak{gl}_d(\mathbb{K})$-equivariant;
- **C)** if $n > 1$ then

$$U_n(v_1, v_2, \dots, v_n) = 0 \tag{19}$$

  for every set of vector fields $v_1, v_2, \dots, v_n \in \mathrm{Der}(A_{\mathrm{formal}})$;
- **D)** if $n \geq 2$ and $v \in \mathrm{Der}(A_{\mathrm{formal}})$ has the form

$$v = \sum_{i,j=1}^{d} v^i_j x^j \frac{\partial}{\partial x^i}, \qquad v^i_j \in \mathbb{K}$$

  then for every set $w_2, \dots, w_n \in V^{\bullet+1}_{A_{formal}}$

$$U_n(v, w_2, \dots, w_n) = 0\,. \tag{20}$$

In paper [16] it was shown that for every degree zero cocycle $\gamma \in \mathsf{GC}$ the structure maps $\Theta(\gamma)_n$ of the L_∞-derivation $\Theta(\gamma)$ satisfy these properties:

a) $\Theta(\gamma)$ can be viewed as an L_∞-derivation of $V^{\bullet+1}_{A_{\mathrm{formal}}}$ with

$$A_{\mathrm{formal}} = \mathbb{K}[[x^1, \dots, x^d]];$$

b) the structure maps $\Theta(\gamma)_n$ of $\Theta(\gamma)$ are $\mathfrak{gl}_d(\mathbb{K})$-equivariant;

c) if $n > 1$ then

$$\Theta(\gamma)_n(v_1, v_2, \dots, v_n) = 0 \qquad (21)$$

for every set of vector fields $v_1, v_2, \dots, v_n \in \mathrm{Der}(A_{\mathrm{formal}})$;

d) if $n \geq 2$ and $v \in \mathrm{Der}(A_{\mathrm{formal}})$ has the form

$$v = \sum_{i,j=1}^{d} v^i_j x^j \frac{\partial}{\partial x^i}\,, \qquad v^i_j \in \mathbb{K}$$

then for every set $w_2, \dots, w_n \in V^{\bullet+1}_{A_{\mathrm{formal}}}$

$$\Theta(\gamma)_n(v, w_2, \dots, w_n) = 0\,. \qquad (22)$$

Properties **a)** and **b)** are obvious, while properties **c)** and **d)** follow from the fact that each graph in the linear combination $\gamma \in \mathsf{GC}$ has only vertices of valencies ≥ 3.

Using these properties of $\Theta(\gamma)$ together with Theorems 1 and 2 we deduce the main result of this note:

Theorem 5. *Every homotopy class of SFQ's contains a representative which can be used to construct a sequence of L_∞ quasi-isomorphisms connecting the sheaf $V^{\bullet+1}_X$ of polyvector fields to the sheaf $\mathcal{D}^{\bullet+1}_X$ of polydifferential operators on a smooth algebraic variety X over $\mathbb{K}$.*

Proof. Let F' be an SFQ. Our goal is to prove that the homotopy class of F' contains a representative F whose corresponding L_∞ quasi-isomorphism (16) satisfies Properties **A)**–**D)** listed above.

Let us denote by F_K an SFQ whose corresponding L_∞ quasi-isomorphism

$$U_K : V^{\bullet+1}_A \rightsquigarrow C^{\bullet+1}(A) \qquad (23)$$

satisfies Properties **A)**–**D)**. (For example, we can choose the SFQ coming from Kontsevich's construction [2].)

Theorem 2 implies that there exists a degree zero cocycle $\gamma' \in \mathsf{dfGC}$ such that F' is homotopy equivalent to the SFQ

$$\exp(\gamma')\big(F_K\big)\,. \qquad (24)$$

On the other hand, we have isomorphism (10). Therefore, γ' is cohomologous to a cocycle $\gamma \in \mathsf{GC}$ and hence F' is homotopy equivalent to

$$\exp(\gamma)\big(F_K\big)\,. \qquad (25)$$

Since the L_∞-derivation $\Theta(\gamma)$ satisfies Properties **a)–d)** and the L_∞ quasi-isomorphism (23) satisfies Properties **A)–D)**, we conclude that the L_∞ quasi-isomorphism corresponding to the SFQ (25) also satisfies Properties **A)–D)**.

Theorem 5 is proved. □

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On One Result of F. Berezin

Simon Gindikin

To the memory of Felix Berezin

Abstract. We discuss Berezin's observation about the extension of holomorphic discrete series of representations. We also include a few personal reminiscences.

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I was glad to participate in a program dedicated to the 80th birthday of my deceased friend Felix Alexandrovich Berezin whom his friends called just Alik. I recalled many moments from the 25 years which I knew him. I remember very well how I (as an 18 years old undergraduate student) saw him for the first time in January of 1956 at the 2nd conference on functional analysis in Moscow. The first one was before the war and not too many mathematical events were happening at that time in Moscow. Functional analysis was then one of the most fashionable areas of mathematics and the organization of the conference illustrated the special role which Gelfand played in Moscow's mathematical life. I believe that almost all mathematicians in Moscow attended the plenary sessions. The conference opened with a lecture of Landau on quantum physics and was concluded by Gelfand's talk on problems of functional analysis. Between them there were a few (as I understand now!) carefully selected talks of outstanding mathematicians (I recall M. Krein, Kantorovich, Sobolev, Shilov, Naimark, Vishik and a few foreign participants). Among those speakers one looked different. He was a young man, looking like a boy (with pink cheeks and without a tie!). It was Alik Berezin, who was then 25 years old and have not yet received his PhD degree. He delivered the principal talk on the theory of representations and his coauthors were Gelfand, Naimark and Graev. Of course, the choice of Alik as the presenter showed that Gelfand considered him as a leader in representations at this moment.

This was the time when Alik started to work at Moscow University after several years of work at a high school. I started to attend Gelfand's seminar and remember as Gelfand discussed with Alik his course, which he was going to give at the university. He wanted to give a course on the theory of representations for 2 years, planning to cover all aspects of the theory starting from finite and compact groups up to infinite-dimensional representations. Gelfand explained that such a plan was not realistic. Of course, he was right, but probably he did not choose a diplomatic way of explaining it. It was clear that Alik was not happy. At any rate he announced the course for one year and I was one of the few of its permanent participants. We started to discuss some problems. He has just finished his most important work on representations: the radial parts of Laplace operators on symmetric spaces. It was his PhD thesis. The course lived only one semester. Alik started to move in the direction of physics and I attended his first seminar on quantum mechanics. Later we also met often in Dynkin's seminar on Lie groups. The first "edition" of this seminar was very important in the early steps of Alik's mathematical life. Around 1957 it was already a second version of the seminar for mathematicians of my generation, but Alik, as well as Karpelevich and Piatetskii-Shapiro, at some point returned to the seminar and played an active role. I believe that it was one of the best seminars for young people starting their mathematical life.

Alik liked mathematical conversations. He was full of mathematical ideas and was happy to share them. I want mention that he was unhappy if somebody later did not give him credit for suggested ideas. I think we most actively discussed special functions of several variables and explicit calculations of spherical functions. He thought a lot about how one must look on this theory and made several remarkable contributions (his and Karpelevich's explicit formulas on Grassmannians are my favorite). It started to be clear that a reduction to one-dimensional hypergeometric functions is rarely possible. We both did computations for the zonal spherical function for $SL(3;\mathbb{R})$. We received different intermediate expressions through elliptic integrals. Alik lived with his mother and I remember our long talks in his room. He had an interesting view on the basic problem: to find an integral representations of order equal to the rank of zonal functions through elementary functions.

Alik had a strong interest in my work with Karpelevich on the asymptotic of zonal spherical functions. I recall some of his remarks which to me seemed interesting and deep. First, he compared Laplace operators on symmetric spaces with Schrödinger operators for many particles and the possibility to compute explicitly the asymptotics of the zonal spherical functions with "the weakening of interactions on infinity". He mentioned that it was possible to generalize the potentials at the radial parts of the Laplace operators in such a way that only for some special values of the parameters ("multiplicities") they give operators on symmetric spaces but the results about the asymptotics (c-functions, at particular) must be true in the general case. I do not think that Alik worked systematically on this project or broadly promoted it. So when many years later Opdam and Heckman realized such a possibility they probably did not know about Alik's ideas.

When Alik moved to Physics our mathematical contacts decreased, but we talked from time to time. We had permanent social contacts. We had several joint friends, often went together for weekend trips outside Moscow and also for a few longer trips. Alik very much liked conversations, not necessarily mathematical ones. Out of several occasions I remember especially clearly our rafting trip on a small Siberia river Mana after a conference, our long talks when we shared a hotel room during a conference in Minsk, not long time before his death. He thought a lot about life and needed to talk about it with his friends.

I remember very well the telephone call on July 1980 from my friend Galya Korpelervich who just learned from Lyalya, Alik's wife, about Alik's death in Kolyma, at a geological expedition. Lyalya described in her memoirs her days at Magadan when she tried to organize the transportation of Alik's body to Moscow. I remember this story from the other side. At this moment I had to stay permanently at home with my mother who was recovering after a stroke. I recall telephone conversations with Alik's friend Misha at Magadan (8 hours difference in time!). The arrival was postponed many times. Those were the last days before the Moscow Olympic Games. The mood of the communist bosses was depraved by the boycott of the USA as a reaction to the Afghanistan aggression. The situation was nervous; the police stopped private cars and without any reason removed license plates. When all problems at Magadan were solved and we expected the plane with Alik's casket it turned out that on the same day Moscow expected the olympic torch and it was practically impossible to obtain the permission to transport a casket through Moscow. It seemed to me that it was the last stupid, but terrible jolt of the Soviet power whose hostile pressure Alik painfully felt throughout his whole life.

I want to recall my last mathematical contact with Alik Berezin in 1973–74. I remember that he asked me some questions about several complex variables. He invited me to give talks at his seminar about Penrose twistors. I do not remember the exact sequence of events. Around this time his wife Lyalya moved to an apartment very close to the place where I lived. I remember how happy Alik was at the birth of his daughter Natasha (I believe) at 1976. My younger son was 2 years younger than she. We walked a few times together with the children. Several times we walked together from the university to our homes having long conversations about mathematics and life. Once with a big excitement he talked about his new results on quantizations on classical symmetric complex domains. He was very impressed that such quantizations exist for Planck's constant from a half-line plus a finite number of isolated points. I could not estimate the beauty of this fact from the point of view of the quantization, but it turns out that it can be completely stated in the language of the theory of representations. There are holomorphic discrete series of representations of groups of automorphisms of complex symmetric domains. The fact which was discovered by Alik is that for each such group there is a finite number of unitary representations realized at holomorphic functions which do not participate in the Plancherel formula – the extensions of the holomorphic discrete series. For this reason these representations were not discovered earlier. Several years later these extensions started to be very popular after the works of Rossi-Vergne and Wallach, but Alik never received appropriate credit for his discovery from the representations community.

Let us start from the case of $G = SL(2;\mathbb{R})$. We have the upper half-plane $\mathbb{C}_+ = \{z = x+iy, y > 0\}$ where the group G acts by fraction-linear transformations $z \mapsto gz = (az+d)/(cz+d)$. We consider Hilbert spaces of holomorphic functions on $\mathbb{C}_+$ with the norms

$$||f||_\lambda^2 = \int_{\mathbb{C}^+} |f(z)|^2 y^{\lambda-2} dxdy.$$

For $\lambda > 1$ this norm is positive and is invariant relative to the representation

$$f \mapsto f(gz)(cz+d)^{-\lambda}.$$

For natural numbers λ we have just holomorphic discrete series of G. If we consider representations of the universal covering group of G we can take arbitrary $\lambda > 1$. However, there is one more unitary representation of G which in a sense corresponds to $\lambda = 1$. It is realized on the Hardy space of holomorphic functions with L^2-norm on the boundary $\{y = 0\}$; they have L^2-boundary values. This representation does not participate in the decomposition of the regular representation of G and it is the simplest example of an extension of the holomorphic discrete series.

Alik considered the generalization of this situation to the case of classical symmetric complex domains. E. Cartan proved that there are four classical series of domains at $\mathbb{C}^n$ (their groups are classical) and 2 exceptional domains. Alik investigated invariant Hilbert spaces of holomorphic functions on classical domains depending on a parameter λ (he associated it with Planck's constant) and conjectured the existence of such spaces for a half-line and some isolated values of λ explicitly described. He proved this conjecture for 1st and 4th class of classical domains and in the case of the other two classes he could give only a weaker estimate. So his question to me was if I had ideas how to prove his conjecture in these cases as well and, perhaps also for the exceptional two symmetric domains. Alik gave me a manuscript of an almost ready paper [1].

Alik considered, following E. Cartan, the bounded realizations of classical domains of "disc" type in which the isotropy subgroup acts linearly and the focus is on the harmonic analysis of this subgroup. His principal source of tools was the remarkable book of L.K. Hua on classical domains where there were explicitly computed the Bergman and Cauchy-Szegö kernels for classical domains with exact constants. The book contained beautiful explicit formulas in the style of classical analysis and it impressed many mathematicians.

At this point of my mathematical life I knew that in some cases another way of explicit computations is more effective: through the Siegel domain realizations of Piatetskii-Shapiro-multidimensional versions of upper half-planes. Here the principal role is played not by the compact subgroup, but by the maximal solvable one. Using that technology I could compute Bergman and Cauchy-Szegö kernels for all symmetric domains (not only the classical ones) and moreover for all complex homogeneous symmetric domains [2]. Pretty soon I understood that Berezin's conjecture was one such problem and I had the tools ready to prove it, again not only for all symmetric domains, but also generalizing it to all complex homogeneous domains. On Alik's insistence I published a note [3] about this and he made a reference in his publication [1]. I focused on the maximally general case and worry that, as a result, I missed an opportunity to explain the nature of Alik's remarkable observation on relatively simple examples. I will try to fill this gap now with the example of classical domains of 3rd Cartan type.

Let us start with a remark about $SL(2;\mathbb{R})$. Let $||f||^2(y)$ be the usual L^2-norm of a holomorphic function $f(z) = f(x+iy)$, $z \in \mathbb{C}_+$ for a fixed $y > 0$. Then

$$||f||^2_\lambda = \int_0^\infty ||f||^2(y) y^{\lambda-2} dy.$$

So we apply for $\lambda > 1$ the regular positive distribution $y^{\lambda-2}$ to $||f||^2(y)$. If we divide this distribution by $\Gamma(\lambda-1)$ we have a distribution which admits a holomorphic extension on all $\lambda \in \mathbb{C}$. For $\lambda = 1$ we have $\delta(y)$ which is positive yet and gives Hardy norm on holomorphic functions. For other λ the distribution is not positive and we do not have norms.

It turns out that there is a similar situation for all symmetric domains. The classical domains of 3rd type can be realized as Siegel half-planes S_l: manifolds of complex symmetric matrices $Z = X + iY$ of order l with positive imaginary parts Y, if the real symplectic group $Sp(l;\mathbb{R})$ is realized as matrices of order $2l$

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with the blocks of order l such that

$$g^\top J g = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the unit matrix of order l.

The Siegel half-plane S_l is invariant relative to the matrix linear-fractional action of these matrices

$$gZ = (AZ+B)(CZ+D)^{-1}.$$

There is also a holomorphically equivalent bounded realization – the Siegel disk (also in the space of symmetric matrices): $I - Z\bar{Z} >> 0$. Berezin worked with this realization.

We define for holomorphic functions the norms $||f||(Y)$ for fixed Y and

$$||f||^2_\lambda = \int_{S_l} |f(X+iY)|^2 (\det Y)^{\lambda-l-1} dX dY = \int_{Y>>0} ||f||^2 (\det Y)^{\lambda-l-1} dY.$$

For natural numbers λ, that are big enough ($\lambda > (l+1)/2$), these norms are invariant relative to the representation

$$f \mapsto f(gZ)(\det(CZ+D))^{-\lambda}.$$

For arbitrary positive λ we can consider representations of the universal covering of the symplectic group. These are unitary holomorphic discrete series of representations. Let V_l be the convex homogeneous cone of positive symmetric matrices. The group $GL(l;\mathbb{R})$ acts transitively by the transforms $Y \mapsto u^\top Y u$ and

$(\det Y)^{\lambda-(l+1)/2}dY$ is the invariant measure relative to this action. It explains the structure of the weight in the norm. It is possible to show that the product of this function on the characteristic function $\chi(Y)$ of V_l is a regular positive distribution and as in the one-dimensional case it "likes" to be divided by an appropriate generalization of the gamma-function. Siegel introduced such a function for the cone of symmetric matrices V;

$$\Gamma_V(\mu) = \int_V \exp(-tr(Y))(\det Y)^{\mu-(l+1)/2}dY.$$

He found that it can be expressed through the one-dimensional Euler gamma-function:

$$\Gamma_V(\mu) = \pi^{l(l-1)/2} \prod_{1\le i\le l} \Gamma(\mu - (l-i)/2).$$

Siegel computed this integral using orthogonal matrices. My observation was that everything looks much simpler if one uses triangular matrices: the substitution $Y = tt^\top$, where t is an upper triangular matrix with positive diagonal elements, transforms this multidimensional integral in a product of several integrals $\int_{-\infty}^{\infty} \exp(-x^2)dx$ and several one-dimensional Euler integrals Γ- functions. It gave a possibility essentially to generalize Siegel's construction: for all symmetric and convex homogeneous cones and make Γ_V dependent on l parameters. Then the distributions

$$\kappa_V(\mu; Y) = \chi_V(Y)(\det Y)^{\lambda-(l+1)/2}/\Gamma_V(\mu),$$

where χ_V is the characteristic function of the cone V, extends holomorphically on all μ. Again, it is easy to see after the triangular substitution and averaging over non diagonal elements of triangular matrices. Then this distribution transforms to the product

$$c \prod_{1\le i\le l} (s_i)_+^{\mu-(l-i)/2-1}/\Gamma(\mu - (l-i)/2),$$

where s_i are the squares of diagonal elements.

This construction has many interesting applications. This family of distributions depending on the parameter μ is a group relative to the convolution. The convolutions with them are analogues of the Riemann-Liouville operators. For $\mu = 0$ our distribution is the δ-function. Among these convolution operators there are some remarkable differential operators.

For our purpose we need to understand when these distributions are positive. We apply results for one-dimensional distributions. We start from the representations through triangular matrices $Y \in V$ where it is unique. Then we need to investigate it on the boundary V where it exists but is already not unique. We can choose canonical triangular matrices which have zero columns if the diagonal elements are zero.

Of course, κ_V are positive for $\mu > (l-1)/2$. Then the support of the distribution is the closure $cl(V_l)$ of the cone V_l. For $\lambda = l-1$ our distribution on the triangular matrices has support in the matrices with $s_1 = t_{11} = 0$ and at Y-coordinates the support will coincide with the boundary of the cone ∂V_l. Using the $Gl(l;\mathbb{R})$-invariance it easy to see that this distribution coincides with the $\delta(\partial V_l)$-δ-function supported on the boundary of the cone V_l ($\det Y = 0$). Decreasing μ and investigating non unique decompositions of points Y of the boundary of the cone in the product of triangular matrices we can see that the distribution will be positive only for the integer points

$$\mu = \frac{l-1}{2}, \frac{l-2}{2}, \ldots, \frac{1}{2}, 0$$

and our distribution will coincide with the δ-function $\delta(V^j)$ of the submanifolds of matrices of rank not exceeding j, $0 \le j \le l-1$; $V^0 = \{0\}$.

It means that the norms

$$||f||_\lambda / \Gamma_V(\lambda - (l+1)/2),$$

holomorphically extended for the parameter λ, are positive norms for $\lambda > (l+1)/2$ and for

$$\lambda = l - (i-1)/2, 1 \le i \le l.$$

We obtained l norms on the spaces of holomorphic functions which were discovered by Berezin and which extend representations of holomorphic discrete series corresponding to $\lambda > (l+1)/2$. We can interpret these norms as intermediate Bergman-Hardy norms with the integration on invariant submanifolds of the boundary of Siegel half-space: $\operatorname{rank}(\operatorname{Im} Z) \le i$. For $i = 0$ we have the Hardy space with the integration on the edge $\mathbb{R}^{l(l+1)}$ of real symmetric matrices of the tube S_l.

For all complex symmetric domains and, more generally, for all complex homogeneous bounded domains at $\mathbb{C}^n$ all components of these constructions are present, starting with some generalized triangular matrices. In such a way we have shown that the generalized Berezin conjecture is true [3].

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Berezin Quantization on Para-Hermitian Symmetric Spaces

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To the memory of my teacher F.A. Berezin

Abstract. A quantization (symbol calculus) in the spirit of Berezin on para-Hermitian symmetric spaces is constructed

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In this paper we construct a quantization in the spirit of Berezin on para-Hermitian symmetric spaces G/H, we lean on [6], [8], [9].

1. Berezin quantization

Recall the concept of quantization proposed by Berezin, see [1], [2]. We restrict ourselves to a rather simplified version.

Let M be a symplectic manifold. Then $C^\infty(M)$ is a Lie algebra with respect to the Poisson bracket $\{A, B\}$, $A, B \in C^\infty(M)$. Quantization in the sense of Berezin consists of the following two steps.

(I) To construct a family $\mathcal{A}_h$ of associative algebras contained in $C^\infty(M)$ and depending on a parameter $h > 0$ (called the Planck constant), with a multiplication denoted by $*$ (depending on h also). These algebras must satisfy the conditions (a) through (d):
  (a) $\lim_{h\to 0} A_1 * A_2 = A_1 A_2$;
  (b) $\lim_{h\to 0} \frac{i}{h} (A_1 * A_2 - A_2 * A_1) = \{A_1, A_2\}$;
  where the multiplication on the right-hand side of *(a)* is the pointwise multiplication, conditions *(a)* and *(b)* together are called the *correspondence principle* (CP);

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(c) the function $A_0 \equiv 1$ is the unit element of each algebra $\mathcal{A}_h$;
(d) the complex conjugation $A \mapsto \overline{A}$ is an anti-involution of any $\mathcal{A}_h$;

(II) To construct representations $A \mapsto \widehat{A}$ of the algebras $\mathcal{A}_h$ by operators in a Hilbert space.

Berezin mainly considered the case when M is a Hermitian symmetric space G/K. Hence M has a complex structure. Let us realize it as a bounded domain in $\mathbb{C}^m$. The functions in question are functions $A(z,\overline{z})$, $z \in M$, analytic on z and $\overline{z}$ separately. In this case complex conjugation reduces to the permutation of z and $\overline{z}$: $\overline{A(z,\overline{z})} = A(\overline{z}, z)$.

Let $B(z,\overline{z})$ be the Bergman kernel of the domain M. An initial object in the Berezin construction is the so-called *super complete system* (the system of coherent states):

$$\Phi_{\overline{w}}(z) = \Phi(z,\overline{w}) = \Phi_\lambda(z,\overline{w}) = B(z,\overline{w})^{-\lambda/\varkappa},$$

where $\lambda < \lambda_0$ (λ_0 is some number), $\varkappa$ is the genus of the corresponding Jordan algebra. Let $\mathcal{F}_\lambda$ be the Fock space, it is a Hilbert space of analytic functions on M square integrable with respect to the measure $c(\lambda) \cdot B(z,\overline{z})^{\lambda/\varkappa} d\nu(z)$, where $c(\lambda)$ is a normalizing factor (depending on λ analytically), $d\nu(z)$ an invariant measure on M. Let (f_1, f_2) be the inner product in $\mathcal{F}_\lambda$. As a function of z, the function $\Phi_{\overline{w}}(z)$ belongs to $\mathcal{F}_\lambda$ and has the reproducing property:

$$(f, \Phi_{\overline{w}}) = f(w).$$

Let $\widehat{A}$ be a bounded operator on $\mathcal{F}$. Associate to it the function of two variables $z, w \in M$:

$$A(z,\overline{w}) = \frac{1}{\Phi(z,\overline{w})} \left(\widehat{A}\Phi_{\overline{w}}\right)(z).$$

Its restriction to the diagonal, i.e., the function $A(z,\overline{z})$ is a function on M, it is called the *covariant symbol* of the operator $\widehat{A}$. The former function $A(z,\overline{w})$ is recovered from $A(z,\overline{z})$ using analyticity.

The operator $\widehat{A}$ is completely determined by its covariant symbol:

$$(\widehat{A}f)(z) = c(\lambda) \int_M A(z,\overline{w}) \, \frac{\Phi(z,\overline{w})}{\Phi(w,\overline{w})} f(w) \, d\nu(w).$$

The multiplication of operators yields a multiplication of symbols:

$$(A_1 * A_2)(z,\overline{z}) = \int_M A_1(z,\overline{w}) \, A_2(w,\overline{z}) \mathcal{B}_\lambda(z,\overline{z}; w,\overline{w}) \, d\nu(w), \tag{1}$$

where

$$\mathcal{B}_\lambda(z,\overline{z}; w,\overline{w}) = c(\lambda) \, \frac{\Phi(z,\overline{w}) \, \Phi(w,\overline{z})}{\Phi(z,\overline{z}) \, \Phi(w,\overline{w})}.$$

This kernel is called the *Berezin kernel*, the operator $\mathcal{B}_\lambda$ with this kernel is called the *Berezin transform*, it acts on functions on M. Berezin ([2], see also [3]) obtained a remarkable formula expressing the Berezin transform $\mathcal{B}_\lambda$ in terms of the Laplace operators $\Delta_1, \dots, \Delta_r$ on G/K, (generators in the algebra of invariant differential operators on M). This formula implies

$$\mathcal{B}_\lambda \sim 1 - \frac{1}{\lambda}\,\Delta, \quad \lambda \to -\infty, \tag{2}$$

where Δ is the Laplace–Beltrami operator on M. Thus, quantization on $M = G/K$ is completed: as the Planck constant, one has to take $h = -1/\lambda$, algebras $\mathcal{A}_h$ consist of covariant symbols of bounded operators on the Fock space $\mathcal{F}_\lambda$ with the multiplication (1), the asymptotic (2) implies that CP holds.

Besides it, Berezin introduces *contravariant symbols*: a function $F(z, \overline{z})$ on M is called the contravariant symbol of a Toeplitz type operator $\widehat{A}$ defined by

$$\left(\widehat{A}f\right)(z) = c(\lambda) \int_M F(w, \overline{w})\, \frac{\Phi(z, \overline{w})}{\Phi(w, \overline{w})}\, f(w)\, d\nu(w).$$

It turns out that the passage from the contravariant symbol to the covariant symbol of the same operator is given just by the Berezin transform.

2. Para-Hermitian symmetric spaces

Let G/H be a *semisimple symmetric space*. Here G is a connected semisimple Lie group with an involutive automorphism $\sigma \neq 1$, and H is an open subgroup of G^σ, the subgroup of fixed points of σ. We consider that groups act on their homogeneous spaces *from the right*, so that G/H consists of right cosets Hg. There exists a Cartan involution τ of G commuting with σ. Set $K = G^\tau$.

Let $\mathfrak{g}$ be the Lie algebra of G and $B_\mathfrak{g}$ its Killing form. Automorphisms of $\mathfrak{g}$ generated by automorphisms of G are denoted by the same letters. There is a decomposition of $\mathfrak{g}$ into direct sums of $+1$, -1-eigenspaces of σ and τ: $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Subspaces $\mathfrak{h}$ are $\mathfrak{k}$ the Lie algebras of H and K respectively. The subspace $\mathfrak{q}$ is invariant with respect to H and $\mathfrak{h}$ in the adjoint representation. It can be identified with the tangent space to G/H at the point $x^0 = He$. The rank r of G/H is the dimension of Cartan subspaces of $\mathfrak{q}$ (maximal Abelian subalgebras in $\mathfrak{q}$ consisting of semisimple elements).

Now let G/H be a *symplectic* manifold. Then $\mathfrak{h}$ has a non-trivial center $Z(\mathfrak{h})$. For simplicity we assume that G/H is an orbit $\operatorname{Ad} G \cdot Z_0$ of an element $Z_0 \in \mathfrak{g}$. In particular, then $Z_0 \in Z(\mathfrak{h})$.

Further, we can also assume that G is *simple*. Such spaces G/H are divided [4] into four classes: (a) Hermitian symmetric spaces; (b) semi-Kählerian symmetric spaces; (c) para-Hermitian symmetric spaces; (d) complexifications of class (a) spaces. Spaces of class (a) are Riemannian, of other three classes are pseudo-Riemannian (not Riemannian).

We focus on class (c). Here $\dim Z(\mathfrak{h}) = 1$, so that $Z(\mathfrak{h}) = \mathbb{R}Z_0$, and Z_0 can be normalized so that the operator $I = (\operatorname{ad} Z_0)_\mathfrak{q}$ on $\mathfrak{q}$ has eigenvalues ± 1. Therefore, $Z_0 \in \mathfrak{p} \cap \mathfrak{h}$. A symplectic structure on G/H is defined by the bilinear form $\omega(X, Y) = B_\mathfrak{g}(X, IY)$ on $\mathfrak{q}$. The ± 1-eigenspaces $\mathfrak{q}^\pm \subset \mathfrak{q}$ of I are Lagrangian,

H-invariant, and irreducible. They are Abelian subalgebras of $\mathfrak{g}$. So $\mathfrak{g}$ becomes a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+ \, (= \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_{+1}).$$

Let us introduce a character $h \mapsto b(h)$ of the group H:

$$b(h) = \det(\mathrm{Ad}\, h)|_{\mathfrak{q}^+}.$$

The pair $(\mathfrak{q}^+, \mathfrak{q}^-)$ is a Jordan pair [5] with multiplication

$$\{XYZ\} = (1/2)\,[[X,Y],\, Z].$$

Let $\varkappa$ be its genus. Ranks of $(\mathfrak{q}^+, \mathfrak{q}^-)$, G/H and $K/K\cap H$ coincide (so that in particular G/H has a discrete series).

Set $Q^\pm = \exp \mathfrak{q}^\pm$. The subgroups $P^\pm = HQ^\pm = Q^\pm H$ are maximal parabolic subgroups of G, with H as a Levi subgroup. One has the following decompositions:

$$G = Q^+HK, \quad G = Q^-HK. \tag{3}$$

Let us call them the *Iwasawa* and the *anti-Iwasawa* decomposition (allowing some slang). For an element in G the first factors in right-hand sides in (3) are defined uniquely, whereas the second and the third factors are defined up to an element of $K \cap H$.

The space $S = K/K\cap H$ is a compact manifold. Decompositions (3) give two actions $s \mapsto \widetilde{s}$ and $s \mapsto \widehat{s}$ of G on S. Namely, let $s = s^0k$ where $s^0 = (K\cap H)e$ is the basic point; decompose kg, $g \in G$, in accordance with (3):

$$kg = \exp Y \cdot \widetilde{h} \cdot \widetilde{k}, \quad kg = \exp X \cdot \widehat{h} \cdot \widehat{k}; \tag{4}$$

then $\widetilde{s} = s^0\widetilde{k}$, $\widehat{s} = s^0\widehat{k}$. Thus, the group G acts on $S \times S$ by $(s,t) \mapsto (\widetilde{s}, \widehat{t})$. Writing $\widetilde{s} = s \cdot g$ we get $\widehat{s} = s \cdot \tau(g)$. The stabilizer of the point (s^0, s^0) is H, so that we obtain an equivariant embedding

$$G/H \hookrightarrow S \times S. \tag{5}$$

Let us call s, t *horospherical coordinates* on G/H. The image M of (5) is a single open dense orbit. Thus, $S \times S$ is a compactification of G/H. For the G-orbit structure of $S \times S$, see [6]. Note that G/H can be represented as the tangent (or cotangent) bundle of the manifold S.

We now define an important function $\|s,t\|$ on $S \times S$. For $s, t \in S$ take k_s, k_t in K so that $s = s^0k_s$, $t = s^0k_t$, and decompose $k_sk_t^{-1}$ as follows (the Gauss decomposition):

$$k_sk_t^{-1} = \exp Y \cdot h \cdot \exp X, \tag{6}$$

where $Y \in \mathfrak{q}^+$, $X \in \mathfrak{q}^-$. For this h, the character $b(h)$ depends only on s,t, but not on the choice of k_s, k_t. We set

$$\|s,t\| = |b(h)|^{-1/\varkappa}, \tag{7}$$

where h is taken from (6). Formula (7) defines $\|s,t\|$ on an open dense subset of $S \times S$. This function is continuous, symmetric and invariant with respect to the diagonal action of K. It can be expanded on the whole $S \times S$, keeping all these properties. The orbit M is characterized by the condition $\|s,t\| \neq 0$. Let ds be a K-invariant measure on S, then the G-invariant measure on G/H is:

$$dx = dx(s,t) = \|s,t\|^{-\varkappa}\, ds\, dt.$$

3. Maximal degenerate series representations

For $\lambda \in \mathbb{C}$, we take the character of H:

$$\omega_\lambda(h) = |b(h)|^{-\lambda/\varkappa}$$

and extend this character to the subgroups $P^\pm$, setting it equal to 1 on $Q^\pm$. Then we consider induced representations of G:

$$\pi_\lambda^\pm = \mathrm{Ind}_{P^\mp}^G\, \omega_{\mp\lambda}.$$

In the compact picture, these representations act on $\mathcal{D}(S)$ by

$$\left(\pi_\lambda^-(g)\,\varphi\right)(s) = \omega_\lambda(\widetilde{h})\,\varphi(\widetilde{s}),$$
$$\left(\pi_\lambda^+(g)\,\varphi\right)(s) = \omega_\lambda(\widehat{h}^{-1})\,\varphi(\widehat{s}),$$

we use (4); note that $\omega_\lambda(\widetilde{h})$ and $\omega_\lambda(\widehat{h}^{-1})$ are well defined because $\omega_\lambda(l) = 1$ for $l \in K \cap H$. For the same λ, the representations $\pi_\lambda^\pm$ are connected by τ: $\pi_\lambda^\pm = \pi_\lambda^\mp \circ \tau$, so that if τ is an inner automorphism, then π_λ^+ and π_λ^- are equivalent.

Consider the following Hermitian form in $\mathcal{D}(S)$:

$$(\psi, \varphi)_S = \int_S \psi(s)\overline{\varphi(s)}\, ds.$$

This form is G-invariant for the pairs $\left(\pi_\lambda^+,\, \pi^+_{-\overline{\lambda}-\varkappa}\right)$ and $\left(\pi_\lambda^-,\, \pi^-_{-\overline{\lambda}-\varkappa}\right)$. Therefore, for $\mathrm{Re}\,\lambda = -\varkappa/2$ the representations $\pi_\lambda^\pm$ are unitarizable, and we obtain two *continuous series* of unitary representations. In a generic case, $\pi_\lambda^\pm$ are irreducible: the reducibility is possible only for real λ satisfying some integrality conditions.

On $C^\infty(S)$ define the operator A_λ:

$$(A_\lambda \varphi)(s) = \int_S \|s,\, t\|^{-\lambda-\varkappa}\, \varphi(t)\, dt,$$

the integral converges absolutely for $\mathrm{Re}\,\lambda < -\varkappa + 1$ and is extended on λ-plane as a meromorphic function. This operator intertwines $\pi_\lambda^\pm$ with $\pi^\mp_{-\lambda-\varkappa}$:

$$A_\lambda \pi_\lambda^\pm(g) = \pi^\mp_{-\lambda-\varkappa}(g) A_\lambda.$$

Moreover,

$$A_{-\lambda-\varkappa} A_\lambda = c(\lambda)^{-1}\, E, \tag{8}$$

where E is the identity operator and $c(\lambda)$ is a meromorphic function.

We can extend $\pi_\lambda^\pm$ and A_λ to the space $\mathcal{D}'(S)$ of distributions on S.

4. Super complete systems and symbols

In this Section we give main constructions of a quantization in the spirit of Berezin on *para-Hermitian* symmetric spaces G/H. Conditions (a)–(d) from § 1 have to be slightly changed: the factor i in (b) has to be omitted, instead of the complex conjugation of functions one has to take some permutation of arguments, finally, we abandon the Hilbert structure in representation spaces.

In general, there is an analogy between classes (a) and (c) (see § 2). At the coordinate level we have an analogy between coordinates $z, \overline{z}$ from § 1 and horospherical coordinates s, t. For G/H, the role of the Fock space is played by a space of functions $\varphi(s)$ of one of these coordinates, we take the space $\mathcal{D}(S)$. As a super complete system we take the kernel of the intertwining operator from § 3, i.e., the function

$$\Phi(s,t) = \Phi_\lambda(s,t) = \|s,t\|^\lambda.$$

It has the reproducing property, which is formula (8) written in another form:

$$\varphi(s) = c(\lambda) \int_{S\times S} \frac{\Phi(s,v)}{\Phi(u,v)}\, \varphi(u)\, dx(u,v).$$

Let $\widehat{A}$ be an operator acting on functions on S. Define the *covariant symbol* $A(s,t)$ of $\widehat{A}$ as follows:

$$A(s,t) = \frac{(\widehat{A}\otimes 1)\Phi(s,t)}{\Phi(s,t)}.$$

We can regard it as a function $A(x)$ on G/H, using (5). The operator is recovered by its symbol:

$$\left(\widehat{A}\varphi\right)(s) = c(\lambda) \int_{S\times S} A(s,v)\, \frac{\Phi(s,v)}{\Phi(u,v)}\, \varphi(u)\, dx(u,v).$$

The identity operator has 1 as its symbol. The multiplication $\widehat{A}_1\widehat{A}_2$ of operators gives rise to the multiplication $A_1 * A_2$ of the symbols:

$$(A_1 * A_2)(s,t) = \int_{S\times S} A_1(s,v) A_2(u,t)\, \mathcal{B}_\lambda(s,t;u,v)\, dx(u,v), \qquad (9)$$

where

$$\mathcal{B}_\lambda(s,t;u,v) = c(\lambda)\, \frac{\Phi(s,v)\Phi(u,t)}{\Phi(s,t)\Phi(u,v)}\ .$$

Let us call this kernel the *Berezin kernel*. By (5) it can be regarded as a function $\mathcal{B}_\lambda(x,y)$, $x, y \in G/H$.

On the other hand, let $F(s,t)$ be a function on $S\times S$. It gives rise to a Toeplitz type operator $\widehat{A}$ by

$$\left(\widehat{A}\varphi\right)(s) = c(\lambda) \int_{S\times S} F(u,v)\, \frac{\Phi(s,v)}{\Phi(u,v)}\, \varphi(u)\, dx(u,v).$$

Let us call the function $F(s,t)$ *contravariant symbol* of the operator $\widehat{A}$. We get a correspondence chain: $F \mapsto \widehat{A} \mapsto A$. We call the composition $\mathcal{B}_\lambda : F \mapsto A$

the *Berezin transform*. It is defined by the same kernel as the multiplication of covariant symbols:

$$A(s,t) = \int_{S\times S} \mathcal{B}_\lambda(s,t;u,v)\, F(u,v)\, dx(u,v).$$

Thus, we have a method for constructing a family of algebras $\mathcal{A}_h$: they consist of the covariant symbols $A(s,t) = A(x)$ of operators from some class, the multiplication in $\mathcal{A}_h$ is given by (9), the representations by operators are $A \mapsto \widehat{A}$. For the Planck constant we take $h = -1/\lambda$ (with suitable normalizations of measures).

In particular, if for the initial algebra of operators, we take the algebra of operators $\pi_\lambda^-(X)$, where X runs the universal enveloping algebra of $\mathfrak{g}$, then we obtain *polynomial quantization*, see, for example, [7]. Here co- and contravariant symbols turn out to be *polynomials* on $G/H \subset \mathfrak{g}$.

Let $\widehat{A}'$ be the operator conjugated to an operator $\widehat{A}$ with respect to the bilinear form whose kernel is the kernel of A_λ. Then their covariant symbols are connected by the transposition of the arguments: $A'(s,t) = A(t,s)$. The map $A \mapsto A'$ changes the order of the factors: $(A_1 * A_2)' = A_2' * A_1'$, so it is an anti-involution of any $\mathcal{A}_h$.

5. Canonical representations and quantization

The main tool for studying quantization is the so-called *canonical representations* (this term was introduced in [8]). For Hermitian symmetric spaces G/K, these representations were introduced by Vershik, Gelfand, Graev [8] (for the Lobachevsky plane) and Berezin [1], [2] (in classical case). These representations act by translations in functions on G/K and are unitary with respect to some non-local inner product (now called a *Berezin form*).

We define canonical representations of a group G in a more general setting. We give up the condition of unitarity (as too narrow) and let these representations act on sufficiently extensive spaces, in particular, on spaces of distributions. Moreover, we permit also non transitive actions of a group G. Our approach uses the notion of an "overgroup" and consists in the following.

Let G and $\widetilde{G}$ be semisimple Lie groups and G is a spherical subgroup of the "overgroup" $\widetilde{G}$ (i.e., G is the fixed point subgroup of an involution of $\widetilde{G}$). Let $\widetilde{P}$ be a maximal parabolic subgroup of $\widetilde{G}$, let $\widetilde{R}_\lambda$, $\lambda \in \mathbb{C}$, be a series of representations of $\widetilde{G}$ induced by characters of $\widetilde{P}$. They can depend on some discrete parameters, we do not write them. As a rule, representations $\widetilde{R}_\lambda$ are irreducible. They act on a compact manifold Ω (a flag manifold for $\widetilde{G}$).

Restrictions R_λ of $\widetilde{R}_\lambda$ to G are called canonical representations of G. They act on functions on Ω. In general, Ω is not a homogeneous space for G, there are several open G-orbits on Ω. They are semisimple symmetric spaces G/H_i. The manifold Ω is the closure of the union of open orbits. The series of canonical representations R_λ has an intertwining operator Q_λ called the Berezin transform.

One can consider a different version of canonical representations, namely, the restriction of canonical representations in the first sense to some open orbit G/H_i. Both variants deserve to be investigated. The first variant is in some sense more natural. But for quantization we need just the second variant.

Let G/K be a Hermitian symmetric space. Then $\widetilde{G}$ is the complexification $G^{\mathbb{C}}$ of G, a parabolic subgroup $\widetilde{P}$ is such that $G \cap \widetilde{P} = K$. Representations $\widetilde{R}_\lambda$ of $\widetilde{G}$ form a maximal degenerate series, they act on $\mathcal{D}(\Omega)$ where $\Omega = \widetilde{G} \cap \widetilde{P} = U/K$, U a maximal compact subgroup of $\widetilde{G}$. A canonical representations R_λ acts on the space $\mathcal{D}(M)$, see § 1, by translations and preserves the Berezin form, i.e., the form with the Berezin kernel. An explicit computation of the Plancherel measure for the Berezin form just gives explicit expressions of the Berezin transform in terms of Laplace operators.

Now let G/H be a para-Hermitian symmetric space, see § 2. Then $\widetilde{G}$ is the direct product $G \times G$, it contains G as the diagonal $\{(g,g), g \in G\}$. A parabolic subgroup $\widetilde{P}$ consists of pairs (zh, hn), $z \in Q^-$, $h \in H$, $n \in Q^+$. Let $\widetilde{\omega}_\lambda$ be a character of $\widetilde{P}$ equal to $\omega_\lambda(h)$ at these pairs. The representation of $\widetilde{G}$ induced by $\widetilde{\omega}_{-\lambda-\varkappa}$ is denoted by $\widetilde{R}_\lambda$. The restriction R_λ of $\widetilde{R}_\lambda$ to G (a canonical representation) is nothing but the tensor product $\pi^-_{-\lambda-\varkappa} \otimes \pi^+_{-\lambda-\varkappa}$. It acts on $\mathcal{D}(\Omega)$, $\Omega = S \times S$, and preserves the following sesqui-linear form:

$$(\varphi_1, \varphi_2)_\lambda = c(\lambda) \int \varphi_1(s,t)\, \overline{\varphi_2(u,v)}\, \left(\|s,v\| \cdot \|u,t\|\right)^\lambda \, ds\, dt\, du\, dv. \tag{10}$$

An operator Q_λ on $\mathcal{D}(\Omega)$ with the same kernel intertwines R_λ with $R_{-\lambda-\varkappa}$.

Let us restrict R_λ to $\mathcal{D}(M)$, $M = G/H$, see § 2, and define a map $\varphi \mapsto f$ on $\mathcal{D}(M)$ by

$$f(s,t) = \varphi(s,t) \|s,t\|^{\lambda+\varkappa}.$$

It turns R_λ into the representation U by translations on $\mathcal{D}(M)$:

$$\left(U(g)f\right)(s,t) = f(\widetilde{s}, \widehat{t}),$$

the form (10) to the form E_λ with the Berezin kernel (the *Berezin form*) and the operator Q_λ to the Berezin transform $\mathcal{B}_\lambda$.

We can regard the *Berezin function* $\mathcal{B}(x, x^0)$ as a H-invariant distribution on G/H. Suppose that we succeed expanding $\mathcal{B}(x, x^0)$ in terms of spherical functions (distributions) on G/H. This is equivalent to writing a Plancherel formula for E_λ. Then we can write expressions of E_λ in terms of Laplace operators $\Delta_1, \ldots, \Delta_r$ on every single series of representations occurring in $L^2(G/H)$. This gives us information about the behavior of E_λ on this series as $\lambda \to -\infty$, and we can say whether CP is true on this series. The CP is equivalent to the asymptotic relation $\mathcal{B}_\lambda \sim 1 - (1/\lambda)\,\Delta$, where Δ is the Laplace–Beltrami operator.

6. Quantization on rank one spaces

We consider here the spaces G/H, where $G = \mathrm{SL}(n,\mathbb{R})$, $H = \mathrm{GL}\,(n-1,\mathbb{R})$. Now it is more convenient to realize G/H as the orbit of the $n \times n$ matrix $x^0 = \mathrm{diag}\{0,\ldots,0,1\}$ under the action $x \mapsto g^{-1}xg$ of G. Then G/H consists of matrices x of rank one and trace one. It has rank $r = 1$ and genus $\varkappa = n$. The stabilizer H of x^0 consists of matrices $\mathrm{diag}\,\{a,b\}$ where $a \in \mathrm{G}\,L(n-1,\mathbb{R})$, $b = (\det a)^{-1}$.

These spaces G/H exhaust all para-Hermitian symmetric spaces of rank one up to coverings.

Let us take in $\mathbb{R}^n$ the Euclidean inner product $\langle x,y\rangle = x_1y_1 + \cdots + x_ny_n$ and the norm $|x| = \sqrt{\langle x,x\rangle}$. The manifold S is the unit sphere $|s| = 1$ in $\mathbb{R}^n$ with the identification of points $\pm s$, i.e., S is the $(n-1)$-dimensional real projective space. We have $\|s,t\| = |\langle s,t\rangle|$, so that $\Phi(s,t) = |\langle s,t\rangle|^\lambda$. The embedding (5) is given by

$$x = \frac{t's}{\langle t,s\rangle}\,,$$

with $\langle t,s\rangle \neq 0$, the prime denotes matrix transposition. The metric $\mathrm{tr}\,(dx^2)$ on G/H is G-invariant. It generates the Laplace–Beltrami operator Δ and the measure

$$dx = |\langle t,s\rangle|^{-n}\,dt\,ds,$$

where ds is the Euclidean measure on S. The manifold $M = G/H$ and its boundary (a Stiefel manifold) are given by the conditions $\langle s,t\rangle \neq 0$ and $\langle s,t\rangle = 0$ respectively. In terms of matrices the Berezin kernel is:

$$\mathcal{B}_\lambda(x,y) = c(\lambda)\,|\,\mathrm{tr}(xy)|^\lambda,$$

where

$$c(\lambda) = \left\{2^{n+1}\pi^{n-2}\Gamma(-\lambda-n+1)\Gamma(\lambda+1)\left[\cos\left(\lambda+\frac{n}{2}\right)\pi - \cos\frac{n\pi}{2}\right]\right\}^{-1}.$$

The quasi regular representation U of G on G/H decomposes into irreducible unitary representations of two series (for definiteness, let $n \geqslant 3$): the continuous series representations $T_{\sigma,\varepsilon}$, $\sigma = (-n+1)/2 + i\rho$, $\rho \in \mathbb{R}$, $\varepsilon = 0,1$, and the discrete series representations $T_{\sigma(m)}$, $\sigma(m) = (-n+2)/2 + m$, $m = 0,1,2,\ldots$; all with multiplicity 1, see [9]. Let us write the expression of the Berezin transform for $\mathrm{Re}\,\lambda < (-n+1)/2$ in terms of Δ:

$$\mathcal{B}_\lambda = \frac{\Gamma(-\lambda+\sigma)\Gamma(-\lambda-\sigma-n+1)}{\Gamma(-\lambda)\Gamma(-\lambda-n+1)} \times \frac{[1-\cos\lambda\pi]\cdot\left[\sin\left(\lambda+\frac{n}{2}\right) + (-1)^\varepsilon\sin\left(\sigma+\frac{n}{2}\right)\right]}{\sin\lambda\pi\cdot\left[\cos\frac{n\pi}{2} - \cos\left(\lambda+\frac{n}{2}\right)\pi\right]}.$$

The right-hand side should be regarded as a function of $\Delta = \sigma(\sigma+n-1)$. The first fraction behaves as $1 - \lambda^{-1}\Delta$ when $\lambda \to -\infty$. It is just what we need for CP. In the second fraction, the term with $(-1)^\varepsilon$ disappears on the discrete spectrum. So we have CP on the discrete spectrum for n even.

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Remarks on Singular Symplectic Reduction and Quantization of the Angular Moment

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Abstract. A direct algebraic method of symplectic reduction is demonstrated for some singular problems. The problem of quantization of singular surfaces is discussed.

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Keywords. Moment map, invariant polynomials, singular reduction, star product on singular surface.

1. Introduction

The history of the reduction method dates back to works in celestial mechanics, Jacobi simplified the Kepler problem by reducing the number of variables using rotational symmetry. The general method of Meyer-Marsden-Weinstein gives a symplectic reduction of systems with a free group action and constraint. Generally, the group action may be not free and the constraint set need not to be smooth. The singular points are often the most interesting because they have smaller orbits and a larger symmetry. There are several approaches to this situation. An example of algebraic singular symplectic reduction was considered in [1]. The problem of singular symplectic reduction of the angular moment was studied in [2],[3] by geometric methods. The problem of systems with constraints in quantum field theory comes to Dirac [4]. BRST method and Batalin-Vilkovisky-Fradkin's theory are proposed for gauge systems. They are based on rather complicated homological construction with a crowd of ghosts, see surveys [5], [6] and a recent development in [7]. We consider here few simple examples where the method of algebraic singular reduction ends up to a singular Poisson algebraic variety to be quantized.

2. Regular symplectic reduction

Let X be a smooth manifold with a symplectic form ω; the corresponding Poisson bracket is defined in a space (sheaf) of smooth functions A in X by $q(f,g) =$

$\omega^* (df, dg)$, $f, g \in A$ where ω^* is the dual 2-form on the cotangent bundle. Suppose that a Lie group G acts in X preserving the form ω. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be its dual space. Any element $\gamma \in \mathfrak{g}$ acts by a vector field $t(\gamma)$ in X. The form $t(\gamma) \vee \omega$ is closed since G preserves ω ($\vee$ denotes the contraction operation). A *moment map* for the action of G on (X, ω) is a smooth map $J : X \to \mathfrak{g}^*$ such that

$$\mathrm{d}J(\gamma) = t(\gamma) \vee \omega, \ \gamma \in \mathfrak{g}.$$

The moment map is assumed G-equivariant that is $J(g(x)) = \mathrm{ad}\, g(J(x))$ for any $g \in G$. Then the group G acts in the set $Y = J^{-1}(0)$ which is called *constraint locus*. The ideal I generated by the components of J is closed under the bracket q (that is, Y is a first class constraint). A G-action is called *Hamiltonian* if for any $\gamma \in \mathfrak{g}$ there exists a smooth function H_γ in X such that $t(\gamma) \vee \omega = \mathrm{d}H_\gamma$ and the map

$$\mathfrak{g} \to (A, q) ; \ \gamma \mapsto H_\gamma$$

is a Lie algebra homomorphism.

Theorem ((Meyer-Marsden-Weinstein)[8]). *Let (X, ω, G, J) be a symplectic manifold with Hamiltonian action of a compact Lie group G and a moment map J such that the constraint locus $Y = J^{-1}(0)$ is a submanifold. Suppose that G acts freely on Y (that is all stabilizers are trivial). Then the orbit space $X_{\mathrm{red}} = Y/G$ is a manifold, $\pi : Y \to X_{\mathrm{red}}$ is a principal G -bundle, and there is a symplectic form ω_{red} on X_{red} satisfying $i^*(\omega) = \pi^*(\omega_{\mathrm{red}})$ where $i : Y \to X$ is the inclusion map.*

The pair $(X_{\mathrm{red}}, \omega_{\mathrm{red}})$ is called a *symplectic reduction* (symplectic quotient) of (X, ω) with respect to (G, J). The condition of free group action is violated in several important cases where orbits of the group have various dimensions and the local topological structure of orbits is complicated.

3. Singular reduction

In the general case a pure algebraic method reveals main features of the footing geometry at least in the case of a compact group action. One advantage of this method is simplicity of all constructions. Moreover an algebraic symplectic reduction can be quantized in purely algebraic terms.

Let X be a real algebraic variety endowed with a Poisson bracket q defined in the algebra A of rational functions in X. In a more general setting, let (X, O) be a real algebraic scheme with a Poisson biderivation $q : O \otimes O \to O$. An algebraic action of a classical compact group G in X is given such that q is G-invariant. Let $J : X \to \mathfrak{g}^*$ be an algebraic moment map where $\mathfrak{g}^*$ is the dual space to the Lie algebra $\mathfrak{g}$ of G. Let $\{0\}$ be the zero point of $\mathfrak{g}^*$ and $Y = X \times_{\mathfrak{g}^*} \{0\}$. Then (Y, O_Y) is an algebraic subscheme of X and $O_Y = O/\mathcal{I}$, $\mathcal{I}$ denotes the ideal in the sheaf O generated by the sections $J^*(t_i)$ and $t_1, \ldots, t_r \in \mathfrak{g}^*$ are coordinate functions in $\mathfrak{g}$. Let $\mathcal{O}_Y^G$ be the sheaf of all G-invariant sections of O_Y. It is called a sheaf of observables on the constraint locus Y. Suppose that the sheaf O_Y^G is generated by global sections $a_1, \dots, a_m$. Then there is defined a homomorphism of sheaves of $\mathbb{R}$-algebras

$$O(\mathbb{R}^m) \to O_Y^G,\ s_i \mapsto a_i, i = 1, \dots, m$$

where $s_1, \dots, s_m$ are coordinate functions in $\mathbb{R}^m$. If $\mathcal{K}$ is the kernel of this homomorphism, then $O_Y^G \cong O(\mathbb{R}^m)/\mathcal{K}$. The maximal spectrum $X_{\mathrm{red}} \doteq \operatorname{Specm} O_Y^G$ is isomorphic to the zero set of the ideal $\mathcal{K}$ in $\mathbb{R}^m$. This is typically a singular algebraic variety unless the group acts freely. The bracket q can be lifted to a Poisson bracket q_{red} in O_Y^G. We call the singular Poisson space $\left(X_{\mathrm{red}}, O_Y^G, q_{\mathrm{red}}\right)$ *algebraic symplectic reduction* of (X, O, q, G, J). The locus X can be a superspace where a supergroup acts, then O is a sheaf of regular functions of even and odd variables.

An algebraic reduction can be explicitly constructed in several cases. We do not need a Hamiltonian structure of the action.

4. Poisson bracket in singular surfaces

Let $O(\mathbb{R}^m)$ be the sheaf of algebraic (analytic or smooth) functions in $\mathbb{R}^m$ and $\varphi_1, \dots, \varphi_{m-2} \in O(\mathbb{R}^m)$. The surface

$$V = \{s \in \mathbb{R}^m; \varphi_1(s) = \dots = \varphi_{m-2}(s) = 0\}$$

may have singular points. The bilinear operator

$$q(a,b) = \det \begin{pmatrix} \partial_1 a & \partial_2 a & \dots & \partial_m a \\ \partial_1 b & \partial_2 b & \dots & \partial_m b \\ \partial_1 \varphi_1 & \partial_2 \varphi_1 & \dots & \partial_m \varphi_1 \\ \dots & \dots & \dots & \dots \\ \partial_1 \varphi_{m-2} & \partial_2 \varphi_{m-2} & \dots & \partial_m \varphi_{m-2} \end{pmatrix},\ \partial_k a = \partial a / \partial s_k$$

is well defined in the sheaf $O_V = O(\mathbb{R}^m)/(\varphi_1, \dots, \varphi_k)$. This is a skew-symmetric map $O_V \otimes O_V \to O_V$.

Proposition 1. [9] *For arbitrary sections $\varphi_1, \dots, \varphi_{m-2}, \psi$ the operator ψq generates a Poisson bracket in O_V.*

In other words the determinant of the Jacobian matrix defines a bracket that satisfies the Jacobian identity.

Example 1. A space $X = \mathbb{C}^2$ is supplied with the symplectic form $\omega = i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + i\mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2$. Let $J(z) = -\left(k|z_1|^2 + l|z_2|^2\right) + \lambda,\ \lambda > 0$ be a moment map for some relatively prime integers k, l. The group $\mathbf{SO}(2)$ acts in X by $e^{i\theta} \cdot z = \left(e^{ik\theta} z_1, e^{il\theta} z_2\right)$, $\mathfrak{g} = \mathbb{R}$. Any point $(z_1, 0), z_1 \neq 0$ has stabilizer $\mathbb{Z}_k$ and any point $(0, z_2), z_2 \neq 0$ has stabilizer $\mathbb{Z}_l$; other points have trivial stabilizers. The algebra O^G of observables is generated by the polynomials

$$a_1 = |z_1|^2, a_2 = |z_2|^2,\ a_3 = \operatorname{Re} z_1^l \bar{z}_2^k,\ a_4 = \operatorname{Im} z_1^l \bar{z}_2^k$$

with the only relation $a_1^l a_2^k - a_3^2 - a_4^2 = 0$ that is

$$O_Y^G \cong O\left(\mathbb{R}^4\right)/(f, g)$$

where $f(s) = s_1^l s_2^k - s_3^2 - s_4^2$ and $g(s) = ks_1 + ls_2 - \lambda$. The space $\operatorname{Specm} O_Y^G$ is the algebraic surface $X_{\rm red} = \{s \in \mathbb{R}^4; f(s) = g(s) = 0\}$ with two singular points $p_1 = \{s \in X_{\rm red}; s_1 = 0\}, p_2 = \{s \in X_{\rm red}; s_2 = 0\}$. Any Poisson bracket in O_Y^G is equal to hq where h is a regular function and

$$q(a,b) = \det \begin{pmatrix} \partial_1 a & \partial_2 a & \partial_3 a & \partial_4 a \\ \partial_1 b & \partial_2 b & \partial_3 b & \partial_4 b \\ \partial_1 f & \partial_2 f & \partial_3 f & \partial_4 f \\ \partial_1 g & \partial_2 g & \partial_3 g & \partial_4 g \end{pmatrix}$$

where $\partial_k = \partial/\partial s_k$. It is easy to check that the algebraic reduction of the space $\left(\mathbb{R}^4, \omega, \mathbf{SO}(2), J\right)$ is isomorphic to $(X_{\rm red}, q)$.

Example 2. The phase space $T^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$ is supplied with the standard Poisson bracket

$$\{f,g\} = \sum \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{1}$$

The moment map is given by

$$J : \mathbb{R}^n \times \mathbb{R}^n \to \wedge^2 \mathbb{R}^n$$

where $J(q,p) = q \times p$. The action of the orthogonal group $\boldsymbol{O}(n)$ in $\mathbb{R}^n \times \mathbb{R}^n$ $(q,p) \mapsto (Uq, Up)$, $U \in \mathbf{O}(n)$ preserves the Poisson bracket and commutes with the moment map: $J(Uq, Up) = \wedge^2 U J(q,p)$ where $\wedge^2 U$ denotes the action of the group in its Lie algebra $\mathfrak{o}(n) \cong \wedge^2 \mathbb{R}^n$. This means that $J(e_{jk}) = q_j p_k - q_k p_j$ for elements $e_{jk} = q_j \partial_k - q_k \partial_j$, $j,k = 1,\ldots,n$ of the Lie algebra. The constraint locus $Y = J^{-1}(0)$ consists of pairs (q,p) such that the vectors q and p are proportional.

Let P be the algebra of all real polynomials in $\mathbb{R}^n$. The algebra of observables is then the subalgebra P^G of all polynomials invariant with respect to the action of the orthogonal group restricted to Y. In the case $n > 2$ the algebra P^G is generated by $a_1 = |q|^2$, $a_2 = |p|^2$, $a_3 = \langle q,p \rangle$ with no syzygy, that is $P^G \cong \mathbb{R}[s_1, s_2, s_3]$. The restriction P_Y^G of the algebra P^G to Y has kernel generated by the equation $f(a) \doteq a_3^2 - a_1 a_2 = 0$ which defines a quadratic cone $V = \{s; f(s) = 0\}$. This implies that $P_Y^G \cong \mathbb{R}[s_1, s_2, s_3]/(f)$. A Poisson bracket Q in the algebra P_Y^G of observables is obtained by the calculation of the brackets (1) for invariant polynomials

$$\{a_1, a_2\} = 4a_3, \{a_1, a_3\} = 2a_1, \{a_2, a_3\} = -2a_2$$

which yields

$$Q(a,b) = 2\det \begin{pmatrix} \partial_1 a & \partial_2 a & \partial_3 a \\ \partial_1 b & \partial_2 b & \partial_3 b \\ \partial_1 f & \partial_2 f & \partial_3 f \end{pmatrix} = 2\det \begin{pmatrix} \nabla a \\ \nabla b \\ \nabla f \end{pmatrix}$$

where $a, b \in \mathbb{R}[s_1, s_2, s_3]$ are arbitrary polynomials. The biderivation Q generates a Poisson bracket in the algebra P_Y^G since $Q(fa, b) = Q(a, fb) = fQ(a,b)$.

5. Commuting matrices

Example 3. Let $\mathbb{S}$ be the space of symmetric $n \times n$-matrices with real entries. The cotangent bundle is $T^*(\mathbb{S}) = \mathbb{S} \times \mathbb{S}$ with the Poisson bracket

$$q(f,g) = \sum_{i,j=1}^{n} \frac{\partial f}{\partial a_{ij}} \frac{\partial g}{\partial b_{ij}} - \frac{\partial f}{\partial b_{ij}} \frac{\partial g}{\partial a_{ij}}$$

where $A = \{a_{ij}\}, B = \{b_{ij}\}$ and (A, B) is a point of the space

$$\mathbb{S} \times \mathbb{S} \ (a_{ji} = a_{ij}, b_{ji} = b_{ij}).$$

Let $\mathbf{SO}(n)$ act by conjugation on $\mathbb{S}$ and on the cotangent bundle $\mathbb{S} \times \mathbb{S}$. This action is Hamiltonian with the moment map

$$J : \mathbb{S} \times \mathbb{S} \to \wedge^2 \mathbb{R}^n = \mathfrak{so}(n)^*, \ (A, B) \mapsto [A, B]$$

where $\wedge^2 \mathbb{R}^n$ is identified with the space of antisymmetric matrices. The constraint locus is $Y = \{[A, B] = 0\}$.

Case $n = 2$. The constraint locus is specified as

$$Y = \{(A, B) : a_3 (b_1 - b_2) = b_3 (a_1 - a_2)\}$$

where

$$A = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}, \ B = \begin{pmatrix} b_1 & b_3 \\ b_3 & b_2 \end{pmatrix}.$$

There are five invariant polynomials

$$\alpha_1 = \operatorname{tr} A, \alpha_2 = \det A, \ \beta_1 = \operatorname{tr} B, \beta_2 = \det B, \ \gamma = a_1 b_2 + a_2 b_1 - 2a_3 b_3$$

which generate the algebra P^G of invariant polynomials in $\mathbb{S} \times \mathbb{S}$. Calculating the Poisson bracket

$$q = \frac{\partial}{\partial a_1} \wedge \frac{\partial}{\partial b_1} + \frac{\partial}{\partial a_2} \wedge \frac{\partial}{\partial b_2} + \frac{1}{2} \frac{\partial}{\partial a_3} \wedge \frac{\partial}{\partial b_3}$$

for the invariant polynomials yields

$$\begin{aligned} q_{\text{red}} = {} & 2 \frac{\partial}{\partial \alpha_1} \wedge \frac{\partial}{\partial \beta_1} + \beta_1 \frac{\partial}{\partial \alpha_1} \wedge \frac{\partial}{\partial \beta_2} - \alpha_1 \frac{\partial}{\partial \beta_1} \wedge \frac{\partial}{\partial \alpha_2} + \delta \frac{\partial}{\partial \alpha_2} \wedge \frac{\partial}{\partial \beta_2} \\ & + \left(\alpha_1 \frac{\partial}{\partial \alpha_1} + |A|^2 \frac{\partial}{\partial \alpha_2} - \beta_1 \frac{\partial}{\partial \beta_1} - |B|^2 \frac{\partial}{\partial \beta_2} \right) \wedge \frac{\partial}{\partial \gamma} \end{aligned}$$

where $\delta = \alpha_1 \beta_1 - \gamma = a_2 b_2 + a_1 b_1 + 2a_3 b_3$. The matrix of the form q_{red} has rank 4. The polynomials γ and δ are algebraic over the algebra $S = \mathbb{R}[\alpha_1, \alpha_2, \beta_1, \beta_2]$ since $\delta + \gamma = \alpha_1 \beta_1$ and

$$\gamma^2 - \alpha_1 \beta_1 \gamma + \alpha_2 |B|^2 + \beta_2 |A|^2 = 0. \tag{2}$$

Therefore the algebra P_Y^G is an extension of degree 2 of the free commutative algebra S. The discriminant of this extension is the discriminant of (2)

$$D = \left((a_1 - a_2)^2 + 4\alpha_3^2 \right) \left((b_1 - b_2)^2 + 4b_3^2 \right) = D_A D_B.$$

Here the factor D_A (and D_B) is the discriminant of the characteristic polynomial of A (respectively of B). In geometrical terms, the spectrum of the complexified algebra $P^G \otimes_{\mathbb{R}} \mathbb{C}$ is a two-fold covering of $\mathbb{C}^4$ ramified over the direct product of zero sets of D_A and D_B.

Conclusion 2. *The singular reduction of the variety* $(X, \mathbf{SO}(2), q)$ *of commuting symmetric* 2×2*-matrices restricted to* Y *is a singular hypersurface* $V \subset \mathbb{R}^5$ *defined by equation* (2) *with respect to the variables* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ *with the Poisson bracket* q_{red}.

General case. We have $(n+1)(n+2)/2 - 1$ invariant polynomials $\mathrm{tr}_{k.i}(A, B)$ where

$$\mathrm{tr} \wedge^k (A + \lambda B) = \sum \lambda^i \, \mathrm{tr}_{k.i}(A, B), i = 0, \ldots, k, k = 1, \ldots, n$$
$$\mathrm{tr}_{k,0}(A, B) = \mathrm{tr}_k(A), \mathrm{tr}_{k,k}(A, B) = \mathrm{tr}_k(B).$$

Conjecture. Let Y be the constraint locus for the moment map $J(A, B) = [A, B]$. Then

(1) The algebra P_Y^G is generated by the polynomials $\mathrm{tr}_{k,i}(A, B)$, $1 \le i \le k \le n$ and is an algebraic extension of degree $n!$ of the algebra $S = S_A \otimes S_B$. Here $S_A(S_B)$ is an algebra freely generated by n polynomials $\alpha_k = \mathrm{tr}_k(A)$ $(\beta_k = \mathrm{tr}_k(B))$, $k = 1, 2, \ldots, n$.
(2) The discriminant ideal of the extension $P_Y^G \to S$ is generated by the product $D_A D_B$ where $D_A \in S_A$ is the discriminant of a matrix A written as a polynomial of $\alpha_1, \ldots, \alpha_n$, similarly for D_B.

In geometrical terms this means that the singular reduction of the variety of commuting symmetric $n \times n$-matrices over the field $\mathbb{C}$ is an algebraic variety of dimension $2n$ which is $n!$-fold covering of $\mathbb{C}^n \times \mathbb{C}^n$. The discriminant set is equal to the product of discriminant sets D_A and D_B.

Added in proof: Conjecture (1) was shown to be true. A proof kindly was given by Florian Eisele [10] is based on the Quillen-Suslin theorem on Serre's hypothesis.

6. Deformation quantization of a singular surface

Let f be a polynomial in $\mathbb{R}^3$; the Poisson bracket

$$p_1(a, b) = \det \begin{pmatrix} \nabla a \\ \nabla b \\ \nabla f \end{pmatrix} \doteq \det \begin{pmatrix} \partial_1 a & \partial_2 a & \partial_3 a \\ \partial_1 b & \partial_2 b & \partial_3 b \\ \partial_1 f & \partial_2 f & \partial_3 f \end{pmatrix}$$

is well defined in the polynomial algebra $P = \mathbb{R}[s_1, s_2, s_3]$. A deformation quantization (or star product) of this algebra with the bracket p_1 is a product operation in $P[[t]]$

$$a * b = ab + t p_1(a, b) + t^2 p_2(a, b) + \cdots \tag{3}$$

(t is a formal variable) that is bilinear with respect to the subalgebra $\mathbb{R}[[t]]$ and fulfills the associativity condition

$$(a * b) * c = a * (b * c) . \tag{4}$$

Here $p_2, p_3, \dots$ are bilinear operators in P and p_1 is as above. This operation turns the space $P[[t]]$ in an associative algebra over the algebra $\mathcal{S} \doteq \mathbb{R}[[t]]$ such that $a * b = ab + tp_1(a,b) \bmod (t^2)$ for any $a, b \in P$. The quantization problem is to find bidifferential operators $p_2, p_3, \dots$ defined in P (or in sheaf algebra O) to fulfill the associativity condition in all degrees of t.

On the first step, the problem is to find a solution p_2 to the cohomological equation

$$\begin{aligned} ap_2(b,c) &- p_2(ab,c) + p_2(a,bc) - p_2(a,b)\,c \\ &= p_1(p_1(a,b),c) - p_1(a,p_1(b,c)),\ a,b,c \in P. \end{aligned} \tag{5}$$

The Jacobi identity for the bracket p_1 implies that the right-hand side of the equation is a cocycle.

Proposition 3. *The operator*

$$p_2(a,b) = \frac{1}{2}\det_3 \begin{pmatrix} \nabla^2 a \\ \nabla^2 b \\ \nabla f \otimes \nabla f \end{pmatrix} + \det_3 \begin{pmatrix} \nabla^2 a \\ \nabla f \otimes \nabla b \\ \nabla^2 f \end{pmatrix} + \det_3 \begin{pmatrix} \nabla f \otimes \nabla a \\ \nabla^2 b \\ \nabla^2 f \end{pmatrix} \tag{6}$$

always satisfies (5), *hence the equation* (4) *is fulfilled up to* $O(t^3)$.

7. Next steps

Definition. For an arbitrary $n \geq 2$ we call n-dimensional matrix of order 3 over a field $\mathbf{k}$ an element of the space $\Phi_n = (\mathbf{k}^3)^{\otimes n}$. The factors $\phi_i = \mathbf{k}^3, i = 1, \dots, n$ are called faces of the space Φ_n. We write below a $n+1$-dimensional matrix A in block form

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

where A_1, A_2, A_3 are n-dimensional matrices of order 3 over $\mathbf{k}$,

$$A_p = \{a_{p,i_1,i_2,\dots,i_n}, i_k = 1,2,3\},\ p = 1,2,3.$$

We define the determinant of A to be the $\mathbf{k}$-number

$$\det_{\mathrm{n+1}} A = \sum_{\varepsilon_1,\dots,\varepsilon_n} (-)^{\sigma(\varepsilon_1)+\dots+\sigma(\varepsilon_n)} a_{1,i_1,\dots,i_n} a_{2,j_1,\dots,j_n} a_{3,k_1,\dots,k_n}$$

where ε_q denotes a permutation of the three elements such that $\varepsilon_q(i_q, j_q, k_q) = (1,2,3)$, and $\sigma(\varepsilon_q)$ is the parity of the permutation; $q = 1, \dots, n$. For a smooth function $a : \mathbb{R}^3 \to \mathbb{C}$ and a natural k we consider a differential $\nabla^k a$ as a k-dimensional matrix with entries $(\nabla^k a)_{i_1,\dots,i_k} = \partial^k a / \partial x^{i_1} \dots \partial x^{i_k}$.

Proposition 4. *The star product* (3) *starting with the bracket* p_1 *can be defined by means of bilinear differential operators*

$$p_n(a,b) = \frac{1}{n!}\det_{n+1}\begin{pmatrix} \nabla^n a \\ \nabla^n b \\ (\nabla f)^{\otimes n} \end{pmatrix} + \frac{1}{(n-2)!}\det_{n+1}\begin{pmatrix} \nabla^n a \\ \nabla f \otimes \nabla^{n-1} b \\ \nabla^2 f \otimes (\nabla f)^{\otimes n-1} \end{pmatrix}$$
$$+ \frac{1}{(n-2)!}\det_{n+1}\begin{pmatrix} \nabla f \otimes \nabla^{n-1} a \\ \nabla^n b \\ \nabla^2 f \otimes (\nabla f)^{\otimes n-1} \end{pmatrix} + q_n(a,b) \quad (7)$$

where $q_n(a,b)$ *is a sum of bidifferential operators of order* $(i,j), i+j \leq 2n-2$ *and* $n = 2,3,\ldots,$ $q_2 = 0$.

Here each of the three blocks is a n-dimensional matrix of order 3. In the second term the factor ∇f in the second block belongs to the same face as one of the faces of the tensor $\nabla^2 f$ in the third block. The third term has a similar meaning.

The structure of formula (7) is related to the well-known Kontsevich construction [11] which represents all the terms of p_n of a star product for arbitrary Poisson bracket. Note that (7) provides an explicit evaluation for the highest order coefficients of our construction.

The operators $p_n, n = 2,3,\ldots$ are not defined on the quotient algebra $P_V = P/(f)$ whereas the bracket p_1 is. The construction (7) is a step towards a star product in the algebra $\mathbb{R}[s_1,s_2,s_3]$. With this star product, any quadratic cone $V = \{s; f(x) = \sum a_{ij} s_i s_j = 0\}$ would generate a non-commutative submanifold

$$\mathcal{V} = \{F(s,t) \doteq \sum a_{ij}(t)\, s_i * s_j = 0\}$$

in the space $\operatorname{Spec}\mathbb{R}[s_1,s_2,s_3][[t]$. It will be a quantization of V.

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Duality and the Abel Map for Complex Supercurves

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Abstract. Supercurves are a generalization to supergeometry of Riemann surfaces or algebraic curves. They naturally appear in pairs related by a duality. The super Riemann surfaces appearing as worldsheets in perturbative superstring theory are precisely the self-dual supercurves. I will review known results and open problems in the geometry of supercurves, with a focus on Abel's Theorem.

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1. Introduction

A supercurve is a generalization to supergeometry of the classical notion of an algebraic curve or Riemann surface. In the smooth case, it is a complex supermanifold of dimension $1|1$. Supercurves naturally occur in pairs connected by a duality generalizing in some sense the Serre duality of line bundles on a Riemann surface. The self-dual supercurves are just the super Riemann surfaces studied extensively during the 1980s in connection with superconformal field theories and string theory. General supercurves have additional applications, for example to supersymmetric integrable systems [1].

In this article I review the definitions and basic examples of supercurves, explain how they generalize both Riemann surfaces and super Riemann surfaces, and describe some work in progress on the "super" analogues of classical results about Riemann surfaces. Section 2 gives the definition and two classes of examples: split supercurves, and super elliptic curves. Section 3 introduces divisors and the duality they lead to: supercurves naturally occur in pairs such that the points of one are the irreducible divisors of the other. Section 4 explains contour integration of differentials on supercurves, and the resulting theory of periods, Jacobians and the Abel map. Section 5 is a sketch of work in progress with Mitchell Rothstein, on Abel's Theorem and the Jacobi Inversion Theorem for supercurves. Section 6 mentions some open problems, such as a theory of theta functions for supercurves.

2. Definitions and examples

I will assume general familiarity with both supermanifolds ([2, 3]) and the classical theory of Riemann surfaces ([4, 5]). Fix a complex Grassmann algebra $\Lambda = \mathbb{C}[\beta_1, \beta_2, \dots, \beta_n]$, to be thought of as the supercommutative "ring of constants" over which we are working. For us, a (smooth) supercurve X will be a family of $1|1$-dimensional complex supermanifolds over $\operatorname{Spec} \Lambda = (\mathrm{pt}, \Lambda)$. (More general families are possible, but this already displays the characteristic "super" phenomena and is consistent with the viewpoint of physicists.) That is, X is a Riemann surface X_{red} with a sheaf $\mathcal{O}$ of functions locally isomorphic to $\mathcal{O}_{\mathrm{red}} \otimes \Lambda[\theta]$, where θ is an additional odd generator. More explicitly, the holomorphic functions on an open set U, $\mathcal{O}(U)$, have the form $F(z, \theta) = f(z) + \theta\phi(z)$. Here we show explicitly the dependence on the coordinates z, θ while hiding that on the parameters β_i. This is in keeping with the viewpoint of physicists that z, θ are true (even and odd) variables while the β_i are merely "anticommuting constants".

The global structure of X is described by invertible parity-preserving transition functions on chart overlaps, having the form $\tilde{z} = F(z, \theta)$, $\tilde{\theta} = \Psi(z, \theta)$. Here the reduced part, or "body", of $F(z, \theta)$, namely $f_{\mathrm{red}}(z)$, is the transition function for X_{red} on the same overlap. There is *no* requirement that the transition functions be "superconformal" as there would be for a super Riemann surface.

We view the transition functions as giving the transformation law for Λ-valued points of X. A Λ-valued point in some chart U is a parity-preserving Λ-algebra homomorphism that evaluates functions on U to give elements of Λ. The "constants" β_i must of course evaluate to themselves. Since z and θ are themselves local functions, we give such a homomorphism by first specifying the elements of Λ to which they evaluate, say z_0 and θ_0. The reduced part of z_0 is the coordinate of the underlying reduced point of X_{red}. A general function $G(z, \theta)$ must then evaluate to $G(z_0, \theta_0)$, so a Λ-valued point may indeed be identified with a pair of Λ-valued coordinates (z_0, θ_0) in each chart. When charts overlap, their Λ-valued points are identified if they give the same evaluation of every function on the overlap. This defines a transformation rule of their coordinates (z_0, θ_0), coinciding with the transition functions. Physicists tend to think of supermanifolds in the familiar terms of their Λ-valued points.

The simplest examples of supercurves are the *split* supercurves. To construct one, choose a Riemann surface to serve as X_{red}. Fix some "soul" line bundle $\mathcal{S}$ on X_{red} and define X by transition functions

$$\tilde{z} = f(z), \quad \tilde{\theta} = \theta g(z),$$

where $f(z)$ are transition functions for X_{red} and $g(z)$ are transition functions for $\mathcal{S}$. In effect, X becomes the total space of the dual bundle, with θ as (odd) fiber coordinate. For example, if X_{red} is the complex plane $\mathbb{C}$ and $\mathcal{S}$ is the trivial line bundle, then X is the affine superspace $\mathbb{C}^{1|1}$.

A set of nonsplit examples is provided by super elliptic curves. Fix an even element $\tau \in \Lambda$ with $\operatorname{Im} \tau_{\mathrm{red}} > 0$, and two odd elements $\epsilon, \delta \in \Lambda$. X will be $\mathbb{C}^{1|1}/G$, where the group $G \cong \mathbb{Z} \times \mathbb{Z}$ has generators A, B acting on $\mathbb{C}^{1|1}$ by

$$A(z,\theta) = (z+1,\theta), \quad B(z,\theta) = (z+\tau+\theta\epsilon, \theta+\delta). \tag{1}$$

Then X_{red} is the torus with lattice generated by 1 and τ_{red}. Associated to a supercurve X there is always a split supercurve $X/(\beta_1, \beta_2, \ldots, \beta_n)$, obtained by "setting the β_i equal to zero", and in this case it is the torus with the trivial line bundle on it.

We use these examples to highlight some differences in the behavior of cohomology for ordinary curves and supercurves. For a split supercurve, it is easy to see that the global functions are $H^0(X, \mathcal{O}) = (\mathbb{C}|\Gamma(\mathcal{S})) \otimes \Lambda$. This notation indicates the even and odd subspaces of a super vector space over Λ. That is, the "even functions" of the form $f(z)$ are the even constants from Λ as expected, but there are also "odd" global holomorphic functions $\theta s(z)$ coming from the global sections $s(z)$ of $\mathcal{S}$, if any. Of course, one can take Λ-linear combinations of these, respecting parity, as well. The presence of nonconstant global functions is a counterintuitive but important feature of supergeometry.

For a super elliptic curve, it is not hard to see that global functions are either constants a or of the form $\theta\alpha$ with α constant, but not all of the latter are G-invariant, because of the action $\theta \mapsto \theta + \delta$ of the generator B. In this way one computes that

$$H^0(X, \mathcal{O}) = \{a + \theta\alpha : \alpha\delta = 0\}. \tag{2}$$

Because of the restriction on α, the cohomology is not freely generated as a Λ-module. This is typical for nonsplit supercurves and is a major complication in dealing with them. It means, for example, that there is no simple result like the Riemann-Roch theorem that characterizes cohomology modules by computing their ranks.

Fortunately Serre duality does work for supercurves: $H^1(X, \mathcal{O}) \cong H^0(X, \mathrm{Ber})^*$ as Λ-modules, as shown in [6]. Here the dual space consists of the Λ-linear functionals on $H^0(X, \mathrm{Ber})$. Earlier work had established Serre duality in the sense of $\mathbb{C}$-linear functionals on individual supermanifolds rather than families [7, 8]

Here the dualizing Berezinian or "canonical" sheaf Ber is the line bundle (see Section 4) on X with transition functions

$$\mathrm{ber} \begin{bmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{bmatrix} = \frac{\partial_z F - \partial_z \Psi (\partial_\theta \Psi)^{-1} \partial_\theta F}{\partial_\theta \Psi}. \tag{3}$$

Serre duality is parity-reversing: even elements of $H^1(X, \mathcal{O})$ correspond to odd linear functionals.

In the split case, $\mathrm{Ber} = K\mathcal{S}^{-1}|K$ (we omit the $\otimes\Lambda$ by abuse of notation). That is, the sections of Ber are generated by even sections $f(z)$ of $K\mathcal{S}^{-1}$, where K is the canonical bundle of differentials on X_{red}, and odd sections having the form $\theta s(z)$ with $s(z)$ itself a differential on X_{red}.

In general, $H^0(X, \mathcal{O})$, respectively $H^1(X, \mathcal{O})$, is always a submodule, respectively a quotient, of a free Λ-module. The free modules in question are isomorphic to the cohomologies of the associated split supercurve, and their ranks can be found from the Riemann-Roch theorem applied to $X_{\rm red}$ and $\mathcal{S}$.

The validity of Serre duality for supercurves can be traced to the fact that the Grassmann algebra Λ is a self-injective, or Gorenstein, ring [9, 10]. This means that linear functionals behave almost as nicely as they do on a vector space: any Λ-linear functional on an ideal $I \subset \Lambda$ is given by multiplication by an element of Λ, modulo those elements that annihilate the ideal.

3. Divisors and the dual curve

We use the standard basis for vector fields on a supercurve, $\partial = \partial_z, \;\; D = \partial_\theta + \theta\partial_z$, and observe that $D^2 = \frac{1}{2}[D, D] = \partial$. A divisor on X is a subvariety of dimension $0|1$, locally given by an even equation $G(z, \theta) = 0$ with $G_{\rm red}$ not identically zero. For example, $z - z_0 - \theta\theta_0 = 0$ locally defines a divisor. In general, near a simple zero of $G_{\rm red}$, $G(z, \theta)$ contains a factor $z - z_0 - \theta\theta_0$ with the parameters z_0, θ_0 determined by the conditions

$$G(z_0, \theta_0) = DG(z_0, \theta_0) = 0. \tag{4}$$

This follows from the Taylor series expansion in the form

$$G(z, \theta) = \sum_{j=0}^{\infty} \frac{1}{j!}(z - z_0 - \theta\theta_0)^j [\partial^j G(z_0, \theta_0) + (\theta - \theta_0) D\partial^j G(z_0, \theta_0)]. \tag{5}$$

Although irreducible divisors depend on two parameters (z_0, θ_0) just like Λ-valued points, a crucial observation is that they are *not* points. To see this, we ask how the parameters of the same divisor are related in two overlapping charts. This is easily computed by using the transition functions to write

$$\tilde{z} - \tilde{z}_0 - \tilde{\theta}\tilde{\theta}_0 = F(z, \theta) - \tilde{z}_0 - \Psi(z, \theta)\tilde{\theta}_0, \tag{6}$$

and applying the conditions (4) to this function G to obtain

$$\tilde{z}_0 = F(z_0, \theta_0) + \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}\Psi(z_0, \theta_0), \quad \tilde{\theta}_0 = \frac{DF(z_0, \theta_0)}{D\Psi(z_0, \theta_0)}. \tag{7}$$

Thus the parameters of a divisor have their own transformation rule distinct from that of points. It is automatic that these new transition functions satisfy a cocycle condition and thus they define a new supercurve denoted $\hat{X}$ and called the dual to X. It has the same reduced curve, and due to the symmetry of the function $z - z_0 - \theta\theta_0$ between (z, θ) and (z_0, θ_0), the dual of $\hat{X}$ is necessarily X again. Thus, supercurves naturally occur in pairs, with the points of each representing the irreducible divisors of the other [11]. Not only does either supercurve determine the other, but a chosen atlas on one determines an associated atlas with the same collection of charts on the other.

We easily determine the duals of our basic examples of supercurves. For split X, we find $\hat{X} = (X_{\rm red}, K\mathcal{S}^{-1})$. That is, this duality simply acts as Serre duality on the line bundle characterizing X. The dual of the super elliptic curve X with parameters τ, ϵ, δ is again a super elliptic curve, with parameters $\tau + \epsilon\delta, \delta, \epsilon$. Note in particular the interchange $\epsilon \leftrightarrow \delta$.

Riemann surfaces are special among algebraic varieties in that their irreducible divisors coincide with their points. We have seen that general supercurves do not share this property. The super-analog of a Riemann surface would thus be a self-dual supercurve. These are the "super Riemann surfaces" (also known as superconformal manifolds or SUSY curves) introduced in connection with string theory in the 1980s. From (7) we find that the transition functions of a super Riemann surface are "superconformal", meaning that $DF = \Psi D\Psi$. For split X this means $\mathcal{S}^2 = K$, so that the Serre self-dual line bundle $\mathcal{S}$ defines a spin structure on X_{red}. For super elliptic curves self-duality means $\epsilon = \delta$.

4. Differentials, integration, line bundles

The fundamental exact sequence underlying contour integration theory for supercurves is

$$0 \to \Lambda \to \mathcal{O} \xrightarrow{D} \hat{\mathrm{Ber}} \to 0. \tag{8}$$

It is the analog of the sequence

$$0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} \Omega^1 \to 0 \tag{9}$$

on a Riemann surface. That is, given representatives $F(z,\theta)$ of a function in some local charts on X, one can check that the derivatives $DF(z,\theta)$ transform as local sections of the canonical bundle $\hat{\mathrm{Ber}}$ of the dual curve $\hat{X}$ [following the cosmetic replacement of the arguments (z,θ) by $(\hat{z},\hat{\theta})$]. Sections $\hat{\omega}$ of $\hat{\mathrm{Ber}}$ should be viewed as "holomorphic differentials" on $\hat{X}$, and locally have antiderivatives with respect to D, which are functions on X determined up to a constant. An antiderivative of $f(\hat{z})+\hat{\theta}\phi(\hat{z})$ is $\theta f(z)+\int^z \phi$. Note that integration is parity-reversing, in addition to mapping between a curve and its dual. Once we have local antiderivatives, contour integrals of the form $\int_P^Q \hat{\omega}$ make sense, as follows. If the points P and Q of X lie in a single (contractible) chart, and F is an antiderivative of $\hat{\omega}$ in this chart, then the integral is defined to be $F(Q) - F(P)$. More generally, we define a super contour C as the pair of points P, Q together with a contour from P_{red} to Q_{red} on X_{red}, and we choose a sequence of points $P = P_1, P_2, \ldots, P_k = Q$ along this contour such that each consecutive pair lies in a common chart. Then the contour integral is defined to be

$$\int_C \hat{\omega} = \sum_{i=1}^{k-1} \int_{P_i}^{P_{i+1}} \hat{\omega}. \tag{10}$$

As for Riemann surfaces, this is independent of the choice of intermediate points.

Similarly, periods and residues of a meromorphic differential make sense: the former is the integral around a nontrivial homology cycle (for example, one of the basis A and B cycles) and the latter is the integral around a closed contour encircling a pole. Among the classical facts about Riemann surfaces which generalize to this context, I point out the Riemann bilinear period relation for holomorphic differentials, which here takes the form

$$\sum_{i=1}^{g}[A_i(\omega)B_i(\hat{\omega}) - B_i(\omega)A_i(\hat{\omega})] = 0. \tag{11}$$

Here g is the genus of the (reduced) curve, ω and $\hat{\omega}$ are arbitrary and independent holomorphic differentials on X and $\hat{X}$ respectively, and the notation $A_i(\omega)$ denotes the period of ω around the cycle A_i. On a Riemann surface, this relation is responsible for the symmetry of the period matrix.

As usual, a line bundle on X is defined by even, invertible transition functions $g_{ij}(z,\theta)$ in chart overlaps $U_i \cap U_j$, satisfying a cocycle condition, and line bundles are therefore classified by $H^1(X, \mathcal{O}_{\mathrm{ev}}^{\times})$. The usual exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathrm{ev}} \overset{\exp 2\pi i \cdot}{\longrightarrow} \mathcal{O}_{\mathrm{ev}}^{\times} \to 0 \tag{12}$$

holds, and shows that degree-zero bundles are classified by the component of the Picard group $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_{\mathrm{ev}})/H^1(X,\mathbb{Z})$. By means of Serre and Poincaré duality, this is isomorphic to the Jacobian

$$\mathrm{Jac}(X) = H^0(X, \mathrm{Ber})^*_{\mathrm{odd}}/H_1(X,\mathbb{Z}).$$

This isomorphism is given explicitly by the *Abel map*: a degree-zero bundle on X can be described by the divisor $\sum_a n_a \hat{P}_a$ of a meromorphic section, and corresponds to the odd linear functional on holomorphic differentials (on X) given by

$$\sum_a n_a \int_{\hat{P}_0}^{\hat{P}_a}$$

modulo periods. Here $\sum_a n_a = 0$, and $\hat{P}_0$ is an arbitrary basepoint on $\hat{X}$. Abel's Theorem is due to [12] in the (free) super Riemann surface case, and to [6] in general.

5. Abel's theorem and Jacobi inversion

The classical Abel's Theorem characterizes those divisors of degree zero which are the divisor of some meromorphic function on a Riemann surface. The analog for supercurves was proved in [6] and states that a degree-zero divisor $\Delta = \sum_a n_a \hat{P}_a$ is the divisor of a meromorphic function F if and only if the associated linear functional $\sum_a n_a \int_{\hat{P}_0}^{\hat{P}_a}$ acting on $H^0(X, \mathrm{Ber})$ vanishes modulo periods. That is, the value of this linear functional on any holomorphic differential is equal to the period of the differential around some fixed cycle which is the same for all differentials. Among many classical proofs of Abel's Theorem, that in [5] is based on criteria for the existence of meromorphic differentials with specified poles and residues on X. In order to better understand such criteria in the super case, M. Rothstein and I (work in progress) are adapting this proof to supercurves. A key ingredient is the Riemann reciprocity law generalizing the above bilinear relation:

$$\sum_{i=1}^{g}[A_i(\omega)B_i(\hat{\eta}) - B_i(\omega)A_i(\hat{\eta})] = 2\pi i \sum_a \operatorname{res}_{\hat{P}_a}(\hat{\eta}) \int_{\hat{P}_0}^{\hat{P}_a} \omega. \tag{13}$$

Here ω is a holomorphic differential on X, $\hat{\eta}$ is a meromorphic differential on $\hat{X}$, and the equation holds on the simply-connected interior of the $2g$-sided polygon obtained by cutting X open along the cycles A_i, B_i.

Here is a sketch of the proof of Abel's Theorem in the case of split X, which is technically simplest. The "easy" direction assumes that the divisor Δ is that of a meromorphic function F, in which case we set $2\pi i\hat{\eta} = D\log F$ and apply (13). The right side becomes the Abel map associated to the divisor, and the left side is an integer combination of periods of ω.

For the "hard" direction we have a divisor Δ whose associated linear functional is zero mod periods, and we must construct a meromorphic F with this divisor, which we do by first constructing the differential $2\pi i\hat{\eta}$ which would be $D\log F$. Recall that the sum of residues of a meromorphic differential at all poles vanishes. If $\hat{\eta}$ is such a differential on $\hat{X}$ then so is $\hat{G}\hat{\eta}$ for any holomorphic function $\hat{G}$. The new ingredient in the super case is that $h^0(\hat{\mathcal{S}})$ such nonconstant holomorphic functions do generally exist. Thus, the residues of $\hat{\eta}$ must satisfy $1|h^0(\hat{\mathcal{S}})$ vanishing conditions, which turn out to be sufficient as well as necessary for the existence of such a differential. These conditions can be shown to hold for the differential we seek, because the divisor has degree zero (1 condition) and because the Abel linear functional is assumed to vanish on the holomorphic differentials $D\hat{G}$ ($h^0(\hat{\mathcal{S}})$ conditions). Now that we have a differential with appropriate residues to be $(D\log F)/2\pi i$, its periods can be adjusted to be integers by adding a suitable combination of holomorphic differentials from $H^0(X, \mathrm{Ber})$; we then reconstruct F by integration and exponentiation. This is all as in the classical proof.

We have not completed the proof in the general case, but believe that it presents only technical obstacles. The major complication is that $H^0(X, \mathrm{Ber})$ is not freely generated; in particular it does not have a basis ω_j normalized as in the classical case to have A-periods $A_i(\omega_j) = \delta_{ij}$. One must show that nevertheless there are enough holomorphic differentials to adjust the periods of $\hat{\eta}$ as required in the last step of the proof.

More information about the Abel map is provided by the classical Jacobi Inversion Theorem, which is also the subject of work in progress. The naive super analog would say that every point in the Jacobian of X is the image under the Abel map of a "g-point divisor" having the form $\Delta = \sum_{a=1}^{g}(\hat{P}_a - \hat{P}_0)$. This is not quite true as stated; again we can only sketch the situation in the split case thus far.

Let the points $\hat{P}_a$ have coordinates $(\hat{z}_a, \hat{\theta}_a)$ in some chart. The divisor Δ corresponds to the linear functional that sends the odd holomorphic differentials $\theta\omega_j$ to $\sum_a \int_{\hat{z}_0}^{\hat{z}_a} \omega_j$, and the even differentials s_j to $\sum_a \hat{\theta} s_j(\hat{z})|_{\hat{P}_0}^{\hat{P}_a}$. Given the images of all these differentials, the Jacobi Inversion Problem is to determine the g points

$\hat{P}_a$. In the split case, their even and odd coordinates can be found separately. The classical Jacobi Inversion Theorem determines the $\hat{z}_a$ from the values of $\sum_a \int_{\hat{z}_0}^{\hat{z}_a} \omega_j$. Knowing these, the prescribed values of $\sum_a \hat{\theta} s_j(\hat{z})|_{\hat{P}_0}^{\hat{P}_a}$ give a system of $h^0(\hat{\mathcal{S}})$ linear equations in g unknowns for the $\hat{\theta}_a$. Thus, the divisor is determined uniquely if $h^0(\hat{\mathcal{S}}) = g$ and the coefficient matrix $s_j(\hat{z}_a)$ has maximal rank. The solution is nonunique, and the Abel map has a nontrivial fiber, if $h^0(\hat{\mathcal{S}}) < g$. Finally, if $h^0(\hat{\mathcal{S}}) > g$ one generally needs to allow for more than g points in the divisor Δ.

6. Open problems

Most of the classical theory of Riemann surfaces was extended to super Riemann surfaces during the 1980s, at least under the simplifying assumption that relevant cohomology groups were free modules. Much has now been further extended to general supercurves, and without restriction on the cohomology, but many interesting questions remain open. For lack of space I mention just two.

Can the duality between X and $\hat{X}$ be described explicitly in terms of classical algebraic geometry? That is, if X is given explicitly as the solution set of some polynomial equations in a projective superspace, can the equations of $\hat{X}$ be constructed?

Theta functions for supercurves need to be better understood. Such theta functions exist when the Jacobian is free, and are related to the super tau functions associated to supersymmetric integrable systems [6, 13]. They can also be constructed on super elliptic curves, for example

$$H(z,\theta) = \sum_{n\in\mathbb{Z}} \exp \pi i \left(2nz + n^2\tau + n\theta\epsilon + n^2\theta\epsilon + \frac{1}{3}n^3\delta\epsilon \right) \tag{14}$$

is such a theta function. By this I mean that it is invariant under the A transformation but acquires a phase linear in the coordinates under B:

$$H(z+\tau+\theta\epsilon, \theta+\delta) = H(z,\theta) \exp -\pi i \left(2z + \tau + 2\theta\epsilon + \frac{1}{3}\delta\epsilon \right). \tag{15}$$

One can define a theta subvariety of the Jacobian as the image by the Abel map of $(g-1)$-point divisors. Assuming free cohomology, it would be expected to have codimension $1|0$, making it a true theta divisor, if $h^1(X_{\mathrm{red}}, \mathcal{S}) = g-1$. Its properties are completely unexplored.

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Berezin's Coherent States, Symbols and Transform for Compact Kähler Manifolds

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Abstract. We review coherent state techniques for general quantizable compact Kähler manifolds. Discussed are co- and contravariant Berezin symbols, Berezin-Toeplitz quantization, the Berezin transform, and related natural deformation quantizations (star products). These are the Berezin-Toeplitz, the Berezin, and the Geometric Quantization star product. All three star products exist in this setting and are uniquely defined. They are different, but equivalent. The equivalence transformation is given. Results on the Berezin transform are used in an essential manner.

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1. Introduction

Coherent states are quite well known, wide-spread and extremely useful tools. Their definition depends on the context of the theory and the objects. It is not the intention of this review to give another overview of this huge subject. For this I refer to the existing ones, e.g., see [1], [2].

Coherent states techniques were always one of the important topics of the Białowieża meetings. Berezin contributed in an essential manner to the theory of coherent states on Kähler manifolds [3], [4], [5], [6], [7]. Starting from coherent states he introduced co- and contravariant symbols, the Berezin transform relating these and deduced important results on the deformation quantization (star products) of Kähler manifolds. To be more precise, Berezin only considered certain homogeneous spaces, like certain open domains in $\mathbb{C}^n$, e.g., the unit disk.

Rawnsley, and Cahen, Gutt, and Rawnsley extended these objects to the case of Kähler manifolds which are not necessarily open domains in $\mathbb{C}^n$ [8], [9], [10], [11], [12]. In these cases one needs the existence of a quantum line bundle.

If such a quantum line bundle exists the manifold is called quantizable. In this approach the coherent vectors are parametrized by the elements of the total space of the quantum line bundle. Covariant symbols can be defined. Under restrictive conditions on the manifolds the authors obtain a star product.

In this review we give the definitions and the results for compact quantizable Kähler manifolds without any restriction whatsoever. Starting from a compact Kähler manifold admitting a quantum line bundle we will recall the definition and the results about the Berezin-Toeplitz operator and deformation quantization. We will introduce coherent vectors and states in the spirit of Berezin-Rawnsley. There is only a small modification, as our coherent vectors are parametrized by the elements of the total space of the dual of the quantum line bundle. This has the advantage that taking the semi-classical limit (by considering all tensor powers of the quantum line bundle) will be easier. With the help of the coherent states we will introduce covariant and contravariant symbols, and the Berezin transform relating them. We will present strong asymptotic approximation results for the Berezin transform based on an asymptotic expansion of the Bergman kernel outside the diagonal. In particular, the existence of the Berezin transform (which we will show) gives a way to define what generalizes the Berezin star product also to the case of arbitrary (quantizable) compact Kähler manifolds. We obtain for every quantizable compact Kähler manifold three different star product, the Berezin-Toeplitz $\star_{BT}$, the Berezin $\star_B$, and the star product of geometric quantization $\star_{GQ}$. It turns out that they are all equivalent. We give the equivalence transformations between them. For example, the equivalence between $\star_{BT}$ and $\star_B$ is given by the (formal) Berezin transform. Moreover, the Berezin transform will be helpful to calculate coefficients for the star products.

These results are obtained partly in joint work with M. Bordemann and E. Meinrenken, resp. with Alexander Karabegov [13], [14], [15], [16], [17], [18]. Despite the fact that some of the presented results (suitably modified) are valid also for certain non-compact situations, due to space limitation we will concentrate here from the very beginning on the compact Kähler case.

As far as the basics of the Berezin-Toeplitz quantization technique are concerned see additionally the reviews [19], [20].

2. The geometric setup

We will only consider phase-space manifolds which carry the structure of a compact Kähler manifold (M, ω). Recall that M is a complex manifold (say of complex dimension n) and ω, the Kähler form, is a non-degenerate closed positive $(1,1)$-form.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with point-wise multiplication as associative product. If we forget the complex structure of M, our form ω will become a symplectic form and we can introduce on $C^\infty(M)$ a Lie algebra structure, the *Poisson bracket* $\{.,.\}$, in the following way. First we assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$\omega(X_f,\cdot) = df(\cdot), \qquad \{ f, g \} := \omega(X_f, X_g) \ . \tag{1}$$

In this way $C^\infty(M)$ becomes a *Poisson algebra.*

The next step in the geometric set-up is the choice of a quantum line bundle. In the Kähler case a *quantum line bundle* for (M,ω) is a triple (L,h,∇), where L is a holomorphic line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h and the complex structure, such that the (pre)quantum condition

$$\begin{gathered}\mathrm{curv}_{L,\nabla}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = -\,\mathrm{i}\,\omega(X,Y), \\ \text{in other words} \quad \mathrm{curv}_{L,\nabla} = -\,\mathrm{i}\,\omega\end{gathered} \tag{2}$$

is fulfilled. Note that by the compatibility ∇ is uniquely fixed. In fact, with respect to a local holomorphic frame of the bundle the metric h will be represented by a function $\hat{h}$. In this case the curvature of the bundle is given by $\overline{\partial}\partial\log\hat{h}$ and the quantum condition reads as

$$\mathrm{i}\,\overline{\partial}\partial\log\hat{h} = \omega \ . \tag{3}$$

Remark. Not all Kähler manifolds are quantizable. For example, only those higher-dimensional complex tori are quantizable which admit "enough theta functions", i.e., which are abelian varieties. This is due to the fact, that an important consequence from the quantization condition (2) is that L is a positive line bundle. By the Kodaira embedding theorem there exists a positive tensor power $L^{\otimes m_0}$ which has enough global holomorphic sections to embed the manifold M via these sections into a projective space $\mathbb{P}^N(\mathbb{C})$. Such a bundle $L^{\otimes m_0}$ is called very ample. In the following we will always assume that the quantum bundle L itself is already very ample. This is not a restriction as $L^{\otimes m_0}$ will be a quantum line bundle for the rescaled Kähler form $m_0\omega$.

Next, we consider all positive tensor powers of the quantum line bundle: $(L^m, h^{(m)}, \nabla^{(m)})$, here $L^m := L^{\otimes m}$. We introduce a scalar product on the space of sections. First we take the Liouville form $\Omega = \frac{1}{n!}\omega^{\wedge n}$ as volume form on M and then set for the scalar product and the norm

$$\langle\varphi,\psi\rangle := \int_M h^{(m)}(\varphi,\psi)\,\Omega \ , \qquad ||\varphi|| := \sqrt{\langle\varphi,\varphi\rangle} \ , \tag{4}$$

on the space $\Gamma_\infty(M,L^m)$ of global C^∞-sections. Let $\mathrm{L}^2(M,L^m)$ be the L^2-completed space of sections with respect to this norm. Furthermore, let $\Gamma_{\mathrm{hol}}(M,L^m)$ be the (finite-dimensional) subspace consisting of global holomorphic section and

$$\Pi^{(m)} : \mathrm{L}^2(M,L^m) \to \Gamma_{\mathrm{hol}}(M,L^m) \tag{5}$$

the corresponding orthogonal projection.

3. Berezin-Toeplitz operators

One of the important mathematical aspects of quantization is to replace the classical observable, which is mathematically a function on the phase space, by an operator, which acts on a certain Hilbert space. In the Berezin-Toeplitz (BT) operator quantization this is done like follows.

For a function $f \in C^\infty(M)$ the associated *Toeplitz operator* $T_f^{(m)}$ (*of level* m) is defined by

$$T_f^{(m)} := \Pi^{(m)}\,(f\cdot): \quad \Gamma_{\text{hol}}(M, L^m) \to \Gamma_{\text{hol}}(M, L^m)\ . \tag{6}$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

The space $\Gamma_{\text{hol}}(M, L^m)$ is the quantum space (of level m). The linear map

$$T^{(m)} : C^\infty(M) \to \text{End}\big(\Gamma_{\text{hol}}(M, L^m)\big), \quad f \to T_f^{(m)} = \Pi^{(m)}(f\cdot)\ , m \in \mathbb{N}_0 \tag{7}$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map* (*of level* m). It will neither be a Lie algebra homomorphism nor an associative algebra homomorphism as in general

$$T_f^{(m)}\, T_g^{(m)} = \Pi^{(m)}\,(f\cdot)\,\Pi^{(m)}\,(g\cdot)\,\Pi^{(m)} \neq \Pi^{(m)}\,(fg\cdot)\,\Pi = T_{fg}^{(m)}.$$

As M is a compact Kähler manifold the space $\Gamma_{\text{hol}}(M, L^m)$ is finite-dimensional. Hence, on a fixed level m the BT quantization is a map from the infinite-dimensional commutative algebra of functions to a non-commutative finite-dimensional (matrix) algebra. A lot of classical information will get lost. To recover this information one has to consider not just a single level m but all levels together as done in the

Definition 1. The Berezin-Toeplitz (BT) quantization is the map

$$C^\infty(M) \to \prod_{m \in \mathbb{N}_0} \text{End}(\Gamma_{\text{hol}}(M, L^{(m)})), \qquad f \to (T_f^{(m)})_{m \in \mathbb{N}_0}. \tag{8}$$

In this way a family of finite-dimensional (matrix) algebras and a family of maps are obtained, which in the classical limit should "converges" to the algebra $C^\infty(M)$. That this is indeed the case and what "convergence" means will be made precise in the following.

We denote for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and by

$$||T_f^{(m)}|| := \sup_{\substack{s \in \Gamma_{\text{hol}}(M, L^m) \\ s \neq 0}} \frac{||T_f^{(m)} s||}{||s||} \tag{9}$$

the operator norm with respect to the norm (4) on $\Gamma_{\text{hol}}(M, L^m)$.

The following theorem was shown in 1994.

Theorem 1 (Bordemann, Meinrenken, Schlichenmaier, [13]).

(a) *For every* $f \in C^\infty(M)$ *there exists a* $C > 0$ *such that*

$$|f|_\infty - \frac{C}{m} \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \ . \tag{10}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$.

(b) *For every* $f, g \in C^\infty(M)$

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O\left(\frac{1}{m}\right). \tag{11}$$

(c) *For every* $f, g \in C^\infty(M)$

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O\left(\frac{1}{m}\right) \ . \tag{12}$$

See also [19] for a sketch of the proof. These results can be rephrased that the BT operator quantization has the correct semi-classical limit, or that it is a strict quantization in the sense of Rieffel.

Let us mention that for real-valued f the Toeplitz operator $T_f^{(m)}$ will be selfadjoint. Beside other results from [13] the following will be also useful

Proposition 2. *On every level* m *the Toeplitz map*

$$C^\infty(M) \to \mathrm{End}(\Gamma_{\mathrm{hol}}(M, L^{(m)})), \qquad f \to T_f^{(m)},$$

is surjective.

There exists another quantum operator in the geometric setting, the operator of geometric quantization introduced by Kostant and Souriau. In a first step the prequantum operator associated to the bundle L^m for the function $f \in C^\infty(M)$ is defined as $P_f^{(m)} := \nabla^{(m)}_{X_f^{(m)}} + \mathrm{i}\, f \cdot id$. Here $X_f^{(m)}$ the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$. Next one has to choose a polarization. In general it will not be unique. But in our complex situation there is a canonical one by taking the projection to the space of holomorphic sections. This polarization is called *Kähler polarization.* The operator of geometric quantization is then defined by

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}. \tag{13}$$

By the surjectivity of the Toeplitz map there exists a function f_m, depending on the level m, such that $Q_f^{(m)} = T_{f_m}^{(m)}$. The Tuynman lemma [21] gives

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f}, \tag{14}$$

where Δ is the Laplacian with respect to the Kähler metric given by ω. It should be noted that for (14) the compactness of M is essential.

As a consequence the operators $Q_f^{(m)}$ and the $T_f^{(m)}$ have the same asymptotic behavior.

4. The Berezin-Toeplitz deformation quantization

There is another approach to quantization. One deforms the commutative algebra of functions "into non-commutative directions given by the Poisson bracket". It turns out that this can only be done on the formal level. One obtains a deformation quantization, also called star product. This notion was around quite a long time. In particular, also Berezin approached the quantization of Kähler manifolds from this perspective, see [7], [3], [4], [5], [6]. Finally, the notion was formalized in [22].

Recall that for a given Poisson algebra $(C^\infty(M), \cdot, \{\,,\,\})$ of smooth functions on a manifold M, a *star product* for M is an associative product $\star$ on $C^\infty(M)[[\nu]]$, the space of formal power series with coefficients from $C^\infty(M)$, such that for $f, g \in C^\infty(M)$

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = \mathrm{i}\{f, g\} \mod \nu$.

It can be expressed as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M). \tag{15}$$

It is called differential (or local) if the $C_k(\,,\,)$ are bidifferential operators with respect to their entries.

Two star products $\star$ and $\star'$ for the same Poisson structure are called *equivalent* if and only if there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $B(f) \star' B(g) = B(f \star g)$.

To every equivalence class of a star product its *Deligne-Fedosov class* can be assigned. It is a formal deRham class of the form $cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M, \mathbb{C})[[\nu]])$. This assignment gives a 1:1 correspondence between equivalence classes of star products and such formal forms.

In the Kähler case we might look for star products adapted to the complex structure. Karabegov [23] introduced the notion of star products with *separation of variable type* for differential star products. The star product is of this type if in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in holomorphic and the second argument in anti-holomorphic directions. Bordemann and Waldmann in their construction [24] used the name *star product of Wick type*.[1] All such star products $\star$ are uniquely given by their Karabegov form $kf(\star)$ which is a formal closed $(1,1)$ form.

[1] In Karabegov's original approach the role of holomorphic and antiholomorphic variables are switched, i.e., in the approach of Bordemann-Waldmann they are of anti-Wick type.

Theorem 3 ([13], [15], [25], [17], [14]). *There exists a unique differential star product*

$$f \star_{BT} g = \sum \nu^k C_k(f,g) \tag{16}$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}. \tag{17}$$

This star product is of separation of variables type with classifying Deligne-Fedosov class cl and Karabegov form kf

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}\left(\frac{1}{\nu}[\omega] - \frac{\delta}{2}\right), \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{\mathrm{can}}. \tag{18}$$

First, the asymptotic expansion in (17) has to be understood in a strong operator norm sense. Second, the used forms, resp. classes are defined as follows. Let K_M be the canonical line bundle of M, i.e., the nth exterior power of the holomorphic bundle of 1-differentials. The canonical class δ is the first Chern class of this line bundle, i.e., $\delta := c_1(K_M)$. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{\mathrm{can}}]$.

Using Theorem 1 and the Tuynman relation (14) one can show that there exists a star product $\star_{GQ}$ given by asymptotic expansion of the product of geometric quantization operators. The star product $\star_{GQ}$ is equivalent to $\star_{BT}$, via the equivalence $B(f) := (id - \nu\frac{\Delta}{2})f$. In particular, it has the same Deligne-Fedosov class. But it is not of separation of variable type.

5. The disc bundle

Before we can discuss coherent vectors, states, etc. in our general Kähler manifold setting we have to introduce the disc bundle. Recall that our quantum line bundle L was assumed to be already very ample. We pass to its dual line bundle $(U,k) := (L^*, h^{-1})$ with dual metric k. In the example of the projective space, the quantum line bundle is the hyperplane section bundle and its dual is the tautological line bundle. Inside the total space U, we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda,\lambda) = 1\},$$

and denote by $\tau : Q \to M$ (or $\tau : U \to M$) the projections to the base manifold M.

The bundle Q is a contact manifold, i.e., there is a 1-form ν such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q. Moreover,

$$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M). \tag{19}$$

Denote by $\mathrm{L}^2(Q,\mu)$ the corresponding L^2-space on Q. Let $\mathcal{H}$ be the space of (differentiable) functions on Q which can be extended to holomorphic functions on the disc bundle (i.e., to the "interior" of the circle bundle), and $\mathcal{H}^{(m)}$ the subspace of $\mathcal{H}$ consisting of m-homogeneous functions on Q. Here m-homogeneous means $\psi(c\lambda) = c^m\psi(\lambda)$. For further reference let us introduce the following (orthogonal) projectors: the *Szegö projector* $\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}$, and its components $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, the *Bergman projectors.*

The bundle Q is a S^1-bundle, and the L^m are associated line bundles. The sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m. This identification is given on the level of the L^2 spaces by the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q,\mu), \quad s \mapsto \psi_s \quad \text{where} \tag{20}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))). \tag{21}$$

Restricted to the holomorphic sections we obtain the unitary isomorphism $\gamma_m : \Gamma_{\mathrm{hol}}(M, L^m) \cong \mathcal{H}^{(m)}$.

6. Coherent vectors and states

Let us look again at (21) but now from the point of view of the linear evaluation functional. This means, we fix $\alpha \in U \setminus 0$ and vary the sections s.

The *coherent vector* (*of level* m) associated to the point $\alpha \in U \setminus 0$ is the element $e^{(m)}_\alpha$ of $\Gamma_{\mathrm{hol}}(M, L^m)$ with

$$\langle e^{(m)}_\alpha, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))) \tag{22}$$

for all $s \in \Gamma_{\mathrm{hol}}(M, L^m)$. A direct verification shows $e^{(m)}_{c\alpha} = \bar{c}^m \cdot e^{(m)}_\alpha$ for $c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Moreover, as the bundle is very ample we get $e^{(m)}_\alpha \neq 0$.

Hence the following definition is possible.

The *coherent state* (*of level* m) associated to $x \in M$ is the projective class

$$\mathrm{e}^{(m)}_x := [e^{(m)}_\alpha] \in \mathbb{P}(\Gamma_{\mathrm{hol}}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0. \tag{23}$$

Finally, the *coherent state embedding* is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{\mathrm{hol}}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e^{(m)}_{\tau^{-1}(x)}]. \tag{24}$$

See [26] for some geometric properties of the coherent state embedding.

7. Covariant Berezin symbol

For an operator $A \in \mathrm{End}(\Gamma_{\mathrm{hol}}(M, L^{(m)}))$ its *covariant Berezin symbol* $\sigma^{(m)}(A)$ (*of level* m) is defined as the real-analytic function

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e^{(m)}_\alpha, Ae^{(m)}_\alpha\rangle}{\langle e^{(m)}_\alpha, e^{(m)}_\alpha\rangle}, \quad \alpha \in \tau^{-1}(x) \setminus \{0\}. \tag{25}$$

Using the *coherent projectors*

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \qquad \alpha \in \tau^{-1}(x) \tag{26}$$

it can be rewritten as $\sigma^{(m)}(A) = \mathrm{Tr}(AP_x^{(m)})$.

Under very restrictive conditions on the manifold it is possible to construct the *Berezin star product* with the help of the covariant symbol map. This was done by Berezin himself [5], [6] and later by Cahen, Gutt, and Rawnsley [9], [10], [11], [12] for more examples.

Denote by $\mathcal{A}^{(m)} \le C^\infty(M)$, the subspace of functions which appear as level m covariant symbols of operators. From the surjectivity of the Toeplitz map follows the injectivity of the symbol map (see Section 9). Hence for the two symbols $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed, and we set as deformed product

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B). \tag{27}$$

Now $\star_{(m)}$ defines on $\mathcal{A}^{(m)}$ an associative and non-commutative product. The crucial problem is, how to obtain from $\star_{(m)}$ a star product $\star$ for the functions (or symbols) independent from the level m? In general this is only possible for very limited classes of manifolds.

Using the Berezin transform and its properties discussed in the next section (at least in the case of quantizable compact Kähler manifolds) we will introduce a star product dual to the by Theorem 3 existing $\star_{BT}$. It will generalizes the above star product.

8. Berezin transform

If we start with a function $f \in C^\infty(M)$, take its Toeplitz operator $T_f^{(m)}$, and then calculate the covariant symbol we obtain a map

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}), \tag{28}$$

which we call the *Berezin transform* (*of level* m).

Theorem 4 ([14]). *Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x)\frac{1}{m^i}, \tag{29}$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are maps with $I_0(f) = f, \quad I_1(f) = \Delta f$.

Here Δ is the Laplacian with respect to the metric given by the Kähler form ω. By *complete asymptotic expansion* the following is understood. Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x) \frac{1}{m^i} \right|_\infty \quad \leq \quad \frac{A}{m^r} \ .$$

The proof of this theorem is quite involved. An important intermediate step of independent interest is the off-diagonal asymptotic expansion of the Bergman kernel function in the neighborhood of the diagonal, see [14]. The Bergman projectors $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, were introduced above. They have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha,\beta)$ on $Q \times Q$, i.e.,

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta).$$

In fact they can by expressed with the help of the coherent vectors as

$$\mathcal{B}_m(\alpha,\beta) = \psi_{e_\beta^{(m)}}(\alpha) = \overline{\psi_{e_\alpha^{(m)}}(\beta)} = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle.$$

The Berezin transform can be given as integral over Q

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta). \tag{30}$$

Take $x = \tau(\alpha), y = \tau(\beta)$, $\alpha,\beta \in Q$ and set $u_m(x) := \mathcal{B}_m(\alpha,\alpha)$, $v_m(x,y) := \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)$. These are well-defined functions on M, resp. $M \times M$ and we obtain another description of the Berezin transform now as integral over M

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{u_m(x)} \int_M v_m(x,y)\ f(y)\Omega(y). \tag{31}$$

For more information see [14], or [18] for an overview. Of course, for certain restricted but important non-compact cases the Berezin transform was already introduced and calculated by Berezin. It was a basic tool in his approach to quantization [4]. For other types of non-compact manifolds similar results on the asymptotic expansion of the Berezin transform are also known. See the extensive work of Englis, e.g., [28].

The theorem above has important applications. First, the Property (10) in Theorem 1 is an easy consequence of the existence of the asymptotic expansion of the Berezin transform. Due to place limitations I will skip it and refer only to [16], [14]. Instead we will discuss applications to star products.

8.1. Application: Berezin star products

As promised we will now introduce for general quantizable compact Kähler manifolds the Berezin star product. We extract from the asymptotic expansion of the

Berezin transform (29) the formal expression

$$I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \tag{32}$$

called the *formal Berezin transform*, and set

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)). \tag{33}$$

As $I_0 = id$ this $\star_B$ is a star product for our Kähler manifold, which we call the *Berezin star product.* Obviously, the formal map I gives the equivalence transformation to $\star_{BT}$. Hence, the Deligne-Fedosov classes will be the same. It will be of separation of variable type now but with the role of the variables switched. When the definition with the covariant symbol works (explained in Section 7) it will coincide with the star product defined there.

Let us summarize. By the presented techniques we obtain for quantizable compact Kähler manifolds three different naturally defined star products $\star_{BT}$, $\star_{GQ}$, and $\star_B$. All three are equivalent and have classifying Deligne-Fedosov class

$$cl(\star_{BT}) = cl(\star_B) = cl(\star_{GQ}) = \frac{1}{\mathrm{i}}\left(\frac{1}{\nu}[\omega] - \frac{\delta}{2}\right). \tag{34}$$

But all three are distinct. In fact $\star_{BT}$ is of separation of variables type (Wick-type), $\star_B$ is of separation of variables type with the role of the variables switched (anti-Wick-type), and $\star_{GQ}$ neither. For their Karabegov forms we obtain (see [14], [19])

$$kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{\mathrm{can}}. \qquad kf(\star_B) = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m). \tag{35}$$

The function u_m was introduced above. It is the Bergman kernel evaluated along the diagonal in $Q \times Q$. The symbol $\mathbb{F}(w_m)$ denotes the formal series expansion corresponding to the asymptotic expansion of w_m in terms of $1/m$ for $m \to \infty$ (i.e., if we replace $1/m$ by ν).

In (35) we gave the Karabegov form for the star product in the convention of Karabegov's original definition. Hence $kf(\star_B)$ is the direct definition. For $kf(\star_{BT})$ this should be interpreted as the Karabegov form of the opposite star product $f \star_{BT}^{\mathrm{opp}} g := g \star_{BT} f$. This is a star product with separation of variables in the original Karabegov convention but now for the pseudo-Kähler manifold $(M, -\omega)$. Hence, the minus sign in (35).

Remark. Based on Fedosov's method Bordemann and Waldmann [24] constructed also a unique star product $\star_{BW}$ which is of Wick type. The opposite star product has Karabegov form $kf(\star_{BW}^{\mathrm{opp}}) = (1/\nu)\,\omega$ and it has the same Deligne Fedosov class $cl(\star_{BW})$ as the other star products in (34). This was shown by Karabegov in [29].

8.2. Application: Calculation of the coefficients of the star products

The proof of Theorem 3 gives a recursive definition of the coefficients $C_k(f,g)$. Unfortunately, it is not very constructive. For their calculation the Berezin transform will also be of help. Theorem 4 shows for quantizable compact Kähler manifolds the existence of the asymptotic expansion of the Berezin transform (29). The operators I_i can be expressed (at least in principle) by the asymptotic expansion of expressions formulated in terms of the Bergman kernel. From (29) we get the formal Berezin transform $I = \mathbb{F}(I^{(m)})$ (32). If we know I explicitly we obtain explicitly $\star_B$ by giving the coefficients $C_k^B(f,g)$ of $\star_B$. For this the knowledge of the coefficients $C_k^{BT}(f,g)$ for $\star_{BT}$ will not be needed. All we need is the existence of $\star_{BT}$ to define $\star_B$.

We have to recall from [14] some additional information. The formal Berezin transform I associated to (29), which is defined with the help of the BT operators, was identified in [14] with the formal Berezin transform in the sense of Karabegov [23] associated to the star product dual and opposite to $\star_{BT}$. By its definition (33) it is the Berezin star product $\star_B$. It is a star product of separation of variables type (in the convention of Karabegov).

As it is a differential star product it makes sense to restrict it to open subsets. The formal Berezin transform $I = I_\star$ (associated to a fixed such star product $\star$) is uniquely given by the condition that

$$f \star g = I(g \cdot f) = I(g \star f), \tag{36}$$

for all local functions f, g , f anti-holomorphic, g holomorphic. The last equality is automatic and is due to the fact, that by the separation of variables property $g \star f$ is the point-wise product $g \cdot f$. Taking the formal series for $\star_B$ (15) and for I (32) we get

$$C_k^B(f,g) = I_k(g \cdot f). \tag{37}$$

The C_k can now be obtained by "polarizing" I_k.

In more detail: It was shown by Karabegov, that I_k is a differential operator. In local complex coordinates it has certain derivatives in holomorphic and certain derivatives in anti-holomorphic directions. It is a differential operator of type (k,k). The C_k are bidifferential operators of order $(0,k)$ in the first argument and order $(k,0)$ in the second argument. As f is anti-holomorphic, in I_k it will only see the anti-holomorphic derivatives. The corresponding is true for the holomorphic g.

If we write I_k as summation over multi-indices (i) and (j) we get

$$I_k = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(i)+(j)}}{\partial z_{(i)} \partial \overline{z}_{(j)}}, \quad a^k_{(i),(j)} \in C^\infty(M) \tag{38}$$

and obtain for the coefficient in the star product $\star_B$

$$C_k^B(f,g) = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(j)} f}{\partial \overline{z}_{(j)}} \frac{\partial^{(i)} g}{\partial z_{(i)}}, \tag{39}$$

where the summation is limited by the order condition. Hence, knowing the components I_k of the formal Berezin transform I gives us C_k^B. Moreover, from I we can recursively calculate the coefficients of the inverse I^{-1} as I starts with id. From $f \star_{BT} g = I^{-1}(I(f) \star_B I(g))$, which is the Relation (33) inverted, we can calculate (at least recursively) the coefficients C_k^{BT}. In practice, the recursive calculations turned out to become quite involved.

I like to point out that the chain of arguments was based on the existence of the Berezin transform and its asymptotic expansion for every quantizable compact Kähler manifold. The asymptotic expansion of the Berezin transform itself is again based on the asymptotic off-diagonal expansion of the Bergman kernel. Indeed, the Toeplitz operator can also be expressed via the Bergman kernel. Based on this it is clear that the same procedure will work also work for non-compact manifold cases if we have at least the same (suitably adapted) objects and corresponding results.

In the purely formal star product setting studied by Karabegov [23] the set of star products of separation of variables type, the set of formal Berezin transforms, and the set of formal Karabegov forms are in 1:1 correspondence. Given $I_\star$ the star product $\star$ can be recovered via the correspondence (38) with (39). What generalizes $\star_{BT}$ is the dual and opposite of $\star$.

9. Contravariant symbols

We need Rawnsley's epsilon function to introduce contravariant symbols in the general Kähler manifold setting. It is defined as

$$\epsilon^{(m)} : M \to C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})(x)}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \ \alpha \in \tau^{-1}(x). \tag{40}$$

In the classical homogeneous case considered by Berezin himself the $\epsilon^{(m)}$ was always constant. As $\epsilon^{(m)} > 0$ we introduce the modified measure

$$\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$$

on the space of functions on M.

Given an operator $A \in \mathrm{End}(\Gamma_{\mathrm{hol}}(M, L^{(m)}))$ then **a** contravariant Berezin symbol $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of A is defined by the representation of the operator A as an integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)} \, \Omega_\epsilon^{(m)}(x), \tag{41}$$

if such a representation exists.

As a first result we quote from [19, Prop. 6.8] that the Toeplitz operator $T_f^{(m)}$ admits such a representation with $\check{\sigma}^{(m)}(T_f^{(m)}) = f$. This says, the function f itself is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. Note that the contravariant symbol is not uniquely fixed by the operator.

As an immediate consequence from the surjectivity of the Toeplitz map it follows that every operator A has a contravariant symbol, i.e., every operator A

has a representation (41). We have to keep in mind, that our Kähler manifolds are compact.

Now we introduce on $\mathrm{End}(\Gamma_{\mathrm{hol}}(M, L^{(m)}))$ the Hilbert-Schmidt norm

$$\langle A, C \rangle_{HS} = Tr(A^* \cdot C).$$

In [20], [16] we showed that

$$\langle A, T_f^{(m)} \rangle_{HS} = \langle \sigma^{(m)}(A), f \rangle_\epsilon^{(m)} \ . \tag{42}$$

This says that the Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint. By the adjointness property from the surjectivity of the Toeplitz map the injectivity of the covariant symbol map follows.

As other consequences of the adjointness we get the important results about the trace of the Toeplitz operators (of course related to eigenvalues of $T_f^{(m)}$)

$$\mathrm{tr}(T_f^{(m)}) = \int_M f \ \Omega_\epsilon^{(m)} = \int_M \sigma^{(m)}(T_f^{(m)}) \ \Omega_\epsilon^{(m)}. \tag{43}$$

To show this we have to plug into (42) $A = I$, resp. $f = 1$ and $A = T_f^{(m)^*}$. Moreover from (43) we get for $f = 1$

$$\dim \Gamma_{\mathrm{hol}}(M, L^m) = \int_M \Omega_\epsilon^{(m)} = \int_M \epsilon^{(m)}(x) \ \Omega. \tag{44}$$

In particular, in the special case that $\epsilon^{(m)}(x) = \mathrm{const}$ then

$$\epsilon^{(m)} = \frac{\dim \Gamma_{\mathrm{hol}}(M, L^m)}{\mathrm{vol}_\Omega(M)}. \tag{45}$$

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Physically Acceptable Solutions of an Eigenvalue Equation in Deformation Quantization

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Dedicated to Professor Bogdan Mielnik on the occasion of his 75th birthday

Abstract. Some of the main facts about the representations of states in both the Hilbert space formulation of quantum mechanics and the deformation quantization formulation are recalled. An eigenvalue equation and its solutions in deformation quantization is considered. Criteria of physical acceptability of eigenstates in deformation quantization for systems with phase space $\mathbb{R}^2$ are proposed.

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1. Introduction

Deformation quantization was born in the first half of the previous century as a **physical** theory. E.P. Wigner [1] introduced a quasiprobability distribution. This distribution, known as Wigner function, represents a quantum state of the system. In contrast to the Hilbert space version of quantum mechanics the Wigner function is defined on a classical phase space. Some years later Groenewold [2] and Moyal [3] proposed a $*$-product called the Moyal product, which is an analog of the product of linear operators.

In this way, two basic elements of an alternative approach to quantum mechanics were formulated. The next natural step in the development of deformation quantization as a physical theory would have been to establish application procedures. But, as can be read in reviews on this topic [4], [5], nowadays researchers working on deformation quantization are focusing mainly on mathematical aspects of this theory such as existence of $*$-products or their equivalence.

Hence we decided to return to the physical origin of deformation quantization and to consider the problem of solving an eigenvalue equation. As it can be observed, usually some of solutions of the eigenvalue equation have no physical meaning. Thus it seems to be necessary to establish a procedure for eliminating nonphysical eigenfunctions.

Our contribution is devoted to presenting applicable methods of classifying solutions of eigenvalue equations. We present several criteria which are useful in recognizing nonphysical results. Especially useful seems to be Theorem 4, giving a sufficient and necessary condition for a function to be a Wigner function of a pure state. We consider only the case of systems with phase space $\mathbb{R}^2$. The results can be generalized easily to spaces $\mathbb{R}^{2n},\ n > 1$. Some of presented properties of Wigner functions are valid also for systems with an arbitrary phase space. However, we were not able to find a practical criterion analogous to Theorem 4 for such systems.

This paper is partially based on the more extended work [6].

2. States in quantum mechanics

This section contains a brief review about states in quantum theory. We divide its content into two subsections. The first of them is devoted to the traditional formulation of quantum mechanics in frames of a Hilbert space and linear operators. The latter is focused on quantum states in deformation theory.

2.1. A density operator

As it is widely known [7], [8], in the Hilbert space formulation of quantum mechanics the maximal information about a system is contained in some linear operator called a density operator or a statistical operator. Let **H** denote the Hilbert space of a quantum system. By definition

Definition 1. An operator $\hat{\varrho} : \mathbf{H} \to \mathbf{H}$ is a **density operator**, if it is

(a) self-adjoint,
(b) positively defined, i.e., $\forall_{|\varphi\rangle\in\mathbf{H}}\ \langle\varphi|\,\hat{\varrho}\,|\varphi\rangle\ \geq 0$,
(c) its trace equals one.

Not normalizable density operators can also be considered but in the context of this paper they are beyond our interest.

Time evolution of the density operator $\hat{\varrho}$ is determined by the Liouville-von Neumann equation

$$\frac{\partial\hat{\varrho}(t)}{\partial t} = \frac{1}{i\hbar}[\hat{H}, \hat{\varrho}], \tag{1}$$

where $\hat{H}$ is the Hamilton operator of the system.

The mean value of an observable $\hat{A}$ in a state determined by a density operator $\hat{\varrho}$ is given by the formula

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{A}\hat{\varrho}). \tag{2}$$

A straightforward consequence of this relation is the observation, that the probability of a detection of a quantum system characterized by the density operator $\hat{\varrho}$ in a normalized state $|\varphi\rangle \in \mathbf{H}$, $\langle\varphi|\varphi\rangle = 1$, equals to

$$\mathrm{Tr}\left(|\varphi\rangle\langle\varphi|\,\hat{\varrho}\right). \tag{3}$$

Definition 2. An **eigenvalue equation** for a linear operator $\hat{A} : \mathbf{H} \supset D(\hat{A}) \to \mathbf{H}$ is

$$\hat{A}|\psi_j\rangle = a_j|\psi_j\rangle \ , \ |\psi_j\rangle \in D(\mathbf{H}).$$

Complex numbers $a_j, \ j \in \mathbb{N}$ are called **eigenvalues** and vectors $|\psi_j\rangle$ are **eigenvectors** of the operator $\hat{A}$. By $D(\hat{A})$ we denote a domain of the operator $\hat{A}$.

Since the density operator $\hat{\varrho}$ is self-adjoint, its eigenvalues are real. Moreover, the Hilbert space $\mathbf{H}$ is spanned by the density operator eigenvectors $|\psi_1\rangle, |\psi_2\rangle, \ldots$. As the density operator is positively defined, its eigenvalues are non negative and, from the property $\mathrm{Tr}\,\hat{\varrho} = 1$, their sum equals 1.

Let us denote eigenvalues of the density operator $\hat{\varrho}$ by $p_i, \ i \in \mathbb{N}$. From the formula (3) we see that p_i is the probability of detecting the system in the eigenstate $|\psi_i\rangle$.

Assume that we are interested in calculating the average value of a density operator $\hat{\varrho}_1$ in a state determined by another density operator $\hat{\varrho}_2$. Since eigenvalues of any density operator are non negative numbers, we immediately obtain that

$$\forall\ \hat{\varrho}_1, \hat{\varrho}_2 \ \ \langle \hat{\varrho}_1 \rangle = \mathrm{Tr}\left(\hat{\varrho}_1 \cdot \hat{\varrho}_2\right) \geq 0. \tag{4}$$

Moreover, as every density operator is bounded and its trace equals 1, we can see

$$\forall\ \hat{\varrho}_1, \hat{\varrho}_2 \ \ \mathrm{Tr}\left(\hat{\varrho}_1 \cdot \hat{\varrho}_2^2\right) \leq 1. \tag{5}$$

States of quantum systems are divided in two groups: **pure states**, which are represented by vectors from a Hilbert space $\mathbf{H}$ and **mixed states** which cannot be identified with any direction in the space $\mathbf{H}$. Eigenstates of an operator $\hat{A}$ are by definition pure states.

There exists a convenient criterion to decide if a state described by the statistical operator $\hat{\varrho}$ is pure. Namely the density operator $\hat{\varrho}$ represents a pure state if and only if

$$\hat{\varrho} \cdot \hat{\varrho} = \hat{\varrho} \ \text{ or } \ \mathrm{Tr}\left(\hat{\varrho}^2\right) = 1. \tag{6}$$

We would like to stress that the relations (6) are not of a purely theoretical character and they can be applied in practical considerations.

Detailed analysis of the geometry of pure and mixed states has been done by B. Mielnik in his pioneer work [9].

2.2. Wigner functions for systems in the phase space $\mathbb{R}^2$

For systems in a phase space $\mathbb{R}^2$ isomorphisms between an algebra of functions in the phase space and an algebra of linear operators in a Hilbert space are known (see [10]–[11]). We consider the isomorphism determined by the Weyl ordering.

From the Weyl correspondence [11] we see that the density operator $\hat{\varrho}$ in the phase space $\mathbb{R}^2$ is represented by a function

$$W^{-1}(\hat{\varrho}) = \int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2}|\hat{\varrho}|q + \frac{\xi}{2} \right\rangle \exp\left(\frac{i\xi p}{\hbar}\right) d\xi \tag{7}$$

or equivalently

$$W^{-1}(\hat{\varrho}) = \int_{-\infty}^{+\infty} \left\langle p - \frac{\eta}{2}|\hat{\varrho}|p + \frac{\eta}{2}\right\rangle \exp\left(-\frac{i\eta q}{\hbar}\right) d\eta. \tag{8}$$

The image $W^{-1}(\hat{\varrho})$ of the density operator $\hat{\varrho}$ contains maximal information about the quantum system.

Definition 3. The function $W(q,p) := \frac{1}{2\pi\hbar}W^{-1}(\hat{\varrho})$ represents the state of a quantum system and it is called the **Wigner function**.

Applying this definition we derive several properties of a Wigner function. These properties have been already known (see, e.g., [12]), but we quote them, as they can be used to distinguish between physical and non physical solutions of an eigenvalue equation.

(i) $\int_{\mathbb{R}^2} W(q,p)dqdp = 1$. Indeed, as $\mathrm{Tr}(\hat{\varrho}) = 1$, from (7) we immediately obtain the result.

(ii) A Wigner function $W(q,p)$ is real, i.e., $W(q,p) = \overline{W}(q,p)$. From formula (7)

$$\overline{W}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \overline{\left\langle q - \frac{\xi}{2}|\hat{\varrho}|q + \frac{\xi}{2}\right\rangle} \exp\left(\frac{-i\xi p}{\hbar}\right) d\xi$$
$$\stackrel{(\xi\to-\xi)}{=} \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \left\langle q - \frac{\xi}{2}|\hat{\varrho}^{+}|q + \frac{\xi}{2}\right\rangle \exp\left(\frac{i\xi p}{\hbar}\right) d\xi.$$

The density operator $\hat{\varrho}$ is self-adjoint. Hence we see that the last expression equals $W(q,p)$.

(iii) $\int_{-\infty}^{+\infty} W(q,p)dp$ represents the spatial density of probability.

$$\int_{-\infty}^{+\infty} W(q,p)dp \stackrel{(7)}{=} \langle q|\hat{\varrho}|q\rangle = \mathrm{Tr}(|q\rangle\langle q|\hat{\varrho})$$

which can be interpreted as the spatial density of probability.

(iv) It can be analogously proved that $\int_{-\infty}^{+\infty} W(q,p)dq$ is the density of probability for momentum.

Let us consider a relation between a trace of an operator $\hat{A}$ and the definite integral $\int_{\mathbb{R}^2}$ of the image of this operator in the Weyl correspondence $A(q,p) := W^{-1}(\hat{A})$.

$$\int_{\mathbb{R}^2} A(q,p)dpdq = \int_{\mathbb{R}^2} dpdq \int_{\mathbb{R}} d\eta \left\langle p - \frac{\eta}{2}|\hat{A}|p + \frac{\eta}{2}\right\rangle \exp\left(-\frac{i\eta q}{\hbar}\right)$$
$$= 2\pi\hbar \int_{\mathbb{R}} dp \int_{\mathbb{R}} d\eta\delta(\eta) \left\langle p - \frac{\eta}{2}|\hat{A}|p + \frac{\eta}{2}\right\rangle = 2\pi\hbar \int_{\mathbb{R}} dp \left\langle p|\hat{A}|p\right\rangle = 2\pi\hbar\ \mathrm{Tr}(\hat{A}).$$

Therefore the mean value of the observable $A(q,p)$ equals

$$\langle A(q,p)\rangle = \int_{\mathbb{R}^2} A(q,p) * W(q,p)dpdq. \tag{9}$$

Moreover, since the Moyal product $*$ is closed [12], in fact

(v) $\langle A(q,p)\rangle = \int_{\mathbb{R}^2} A(q,p)\cdot W(q,p)dpdq$.

(vi) The time evolution of a Wigner function is determined by the relation

$$\frac{dW(q,p)}{dt} =, \{H,W\}_M \tag{10}$$

where the symbol $\{H, W(q,p)\}_M := \frac{1}{i\hbar}(H*W - W*H)$ denotes the **Moyal bracket**.

(vii) For two arbitrary Wigner functions

$$\forall_{W_1,W_2} \quad \int_{\mathbb{R}^2} W_1 W_2 dpdq \geq 0. \tag{11}$$

This property follows from the fact that the density operator is positively defined.

(viii) As the density operator is bounded,

$$\begin{aligned} \forall_{W_1,W_2} \quad \int_{\mathbb{R}^2} W_1\big(W_2 * W_2\big)dpdq &= \int_{\mathbb{R}^2} W_2\big(W_1 * W_2\big)dpdq \\ = \int_{\mathbb{R}^2} W_2\big(W_2 * W_1\big)dpdq &\leq \frac{1}{(2\pi\hbar)^2}. \end{aligned} \tag{12}$$

The previous relation also implies

$$\forall_{W_1,W_2} \quad \int_{\mathbb{R}^2} W_2\,\{W_2, W_1\}_M dpdq = 0. \tag{13}$$

Properties of Wigner functions presented above are necessary but not sufficient conditions for functions to be quasiprobability distributions. Thus even if an investigated function satisfies all tested properties, we cannot say that it is definitely a representation of a physical state.

3. Physical solutions of an eigenvalue equation

With the use of the Weyl correspondence we see that an eigenvalue equation for a function A in the phase space is of the form

$$A * W_j = a_j W_j\ , \quad \{A, W_j\}_M = 0 \tag{14}$$

where a_j is an eigenvalue of A assigned to a Wigner eigenfunction W_j (see, e.g., [13]). The eigenfunctions W_j represent pure states. It means that they are images of projective operators in the Weyl correspondence. Thus from (7) we immediately obtain a necessary and sufficient condition for a real function to be a Wigner function of a pure state.

Theorem 1. *A real function $W(q,p)$ defined in the phase space $\mathbb{R}^2$ is a Wigner function of a pure state if and only if*

(a) $\int_{\mathbb{R}^2} dqdp\, W(q,p) = 1$ *and*

(b) $W(q,p) * W(q,p) = \frac{1}{2\pi\hbar} W(q,p)$.

This condition can be applied in an arbitrary phase space. Unfortunately, as the Wigner function contains negative powers of the deformation parameter $\hbar$, in cases when the $*$-product is determined by bidifferential operators, Theorem 1 is hardly applicable.

Only in the phase space $\mathbb{R}^2$, when an integral form of the Moyal product is known, we arrive to the useful conclusion that

Theorem 2. *A necessary and sufficient condition for a real function $W(q,p)$ to represent a pure quantum state is that $\int_{\mathbb{R}^2} dqdp\, W(q,p) = 1$ and*

$$\frac{2}{\pi\hbar}\int_{\mathbb{R}^4} dq'dp'dq''dp''W(q',p')W(q'',p'') \times \exp\left[\frac{2i}{\hbar}\Big\{(q'-q)(p''-p)-(q''-q)(p'-p)\Big\}\right] = W(q,p).$$

Notice that from Theorems 1 or 2 it follows that a necessary condition for a function $W(q,p)$ to be a Wigner function of a pure state is

$$\int_{\mathbb{R}^2} dqdpW^2(q,p) = \frac{1}{2\pi\hbar}. \qquad (15)$$

There exists an elegant and very useful necessary and sufficient condition for a Wigner function to be a Wigner function of a pure state.

Theorem 3 ([12]). *The necessary condition for a Wigner function $W(q,p)$ to represent a pure state is that the function*

$$\varrho(q_1,q_2) := \int_{-\infty}^{+\infty} dpW(p,\frac{q_1+q_2}{2})\exp\left(\frac{ip(q_1-q_2)}{\hbar}\right)$$

satisfies

$$\frac{\partial^2 \ln \varrho(q_1,q_2)}{\partial q_1 \partial q_2} = 0.$$

We were able to modify significantly this theorem to obtain a necessary and sufficient condition for an arbitrary function to be a Wigner function of a pure state (see [6] for detail).

Theorem 4. *A real function $W(q,p)$ defined in the phase space $\mathbb{R}^2$ is a Wigner function of a pure state if and only if*

(a) *at every point $(q,p)\in\mathbb{R}^2$ it is continuous with respect to q and with respect to p,*
(b) $\int_{\mathbb{R}^2} dqdpW(q,p) = 1$,
(c) *for every $q_1,q_2\in\mathbb{R}$ there is $\varrho(q_1,q_2) = f(q_1)g(q_2)$, where*

$$\varrho(q_1,q_2) := \int_{\mathbb{R}} dp\, W\left(\frac{q_1+q_2}{2},p\right)\exp\left[\frac{ip(q_1-q_2)}{\hbar}\right].$$

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A Classification Theorem and a Spectral Sequence for a Locally Free Sheaf Cohomology of a Supermanifold

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To our team coach Yu.A. Kirillov on his 70th birthday

Abstract. This paper is based on the paper [1], where two classification theorems for locally free sheaves on supermanifolds were proved and a spectral sequence for a locally free sheaf of modules $\mathcal{E}$ was obtained. We consider another filtration of the locally free sheaf $\mathcal{E}$, the corresponding classification theorem and the spectral sequence, which is more convenient in some cases. The methods, which we are using here, are similar to [1, 2].

The first spectral sequence of this kind was constructed by A.L. Onishchik in [2] for the tangent sheaf of a supermanifold. However, the spectral sequence considered in this paper is not a generalization of Onishchik's spectral sequence from [2].

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1. Main definitions and classification theorems

1.1. Main definitions

Let $(M, \mathcal{O})$ be a supermanifold of dimension $n|m$, i.e., a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. The underlying complex manifold $(M, \mathcal{F})$ is called the *reduction* of $(M, \mathcal{O})$. The simplest class of supermanifolds constitute the so-called *split supermanifolds*. We recall that a supermanifold $(M, \mathcal{O})$ is called split if $\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is a locally free sheaf of $\mathcal{F}$-modules on M. With any supermanifold $(M, \mathcal{O})$ one can associate a split supermanifold

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$(M,\tilde{\mathcal{O}})$ of the same dimension which is called the *retract* of $(M,\mathcal{O})$. To construct it, let us consider the $\mathbb{Z}_2$-graded sheaf of ideals $\mathcal{J}=\mathcal{J}_{\bar 0}\oplus\mathcal{J}_{\bar 1}\subset\mathcal{O}$ generated by odd elements of $\mathcal{O}$. The structure sheaf of the retract is defined by

$$\tilde{\mathcal{O}}=\bigoplus_{p\geq 0}\tilde{\mathcal{O}}_p,\ \text{ where }\ \tilde{\mathcal{O}}_p=\mathcal{J}^p/\mathcal{J}^{p+1},\ \ \mathcal{J}^0:=\mathcal{O}.$$

Here $\tilde{\mathcal{O}}_1$ is a locally free sheaf of $\mathcal{F}$-modules on M and $\tilde{\mathcal{O}}_p=\bigwedge^p_{\mathcal{F}}\tilde{\mathcal{O}}_1$. By definition, the following sequences

$$\begin{array}{c}0\to\mathcal{J}\cap\mathcal{O}_{\bar 0}\to\mathcal{O}_{\bar 0}\xrightarrow{\pi}\tilde{\mathcal{O}}_0\to 0,\\ 0\to\mathcal{J}^2\cap\mathcal{O}_{\bar 1}\to\mathcal{O}_{\bar 1}\xrightarrow{\tau}\tilde{\mathcal{O}}_1\to 0.\end{array}\tag{1}$$

are exact. Moreover, they are locally split. The supermanifold $(M,\mathcal{O})$ is split iff both sequences are globally split.

Denote by $\mathcal{S}_{\bar 0}$ and $\mathcal{S}_{\bar 1}$ the even and the odd parts of a $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{S}$ on M, respectively; by $\Pi(\mathcal{S})$ we denote the same sheaf of $\mathcal{O}$-modules $\mathcal{S}$ equipped with the following $\mathbb{Z}_2$-grading: $\Pi(\mathcal{S})_{\bar 0}=\mathcal{S}_{\bar 1}$, $\Pi(\mathcal{S})_{\bar 1}=\mathcal{S}_{\bar 0}$. A $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules on M is called *free* (*locally free*) *of rank* $p|q$, $p,q\geq 0$ if it is isomorphic (respectively, locally isomorphic) to the $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{O}^p\oplus\Pi(\mathcal{O})^q$. For example, the tangent sheaf $\mathcal{T}$ of a supermanifold $(M,\mathcal{O})$ is a locally free sheaf of $\mathcal{O}$-modules.

Let now $\mathcal{E}=\mathcal{E}_{\bar 0}\oplus\mathcal{E}_{\bar 1}$ be a locally free sheaf of $\mathcal{O}$-modules of rang $p|q$ on an arbitrary supermanifold $(M,\mathcal{O})$. We are going to construct a locally free sheaf of the same rank on $(M,\tilde{\mathcal{O}})$. First, we note that $\mathcal{E}_{\mathrm{red}}:=\mathcal{E}/\mathcal{J}\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M. Moreover, $\mathcal{E}_{\mathrm{red}}$ admits the $\mathbb{Z}_2$-grading $\mathcal{E}_{\mathrm{red}}=(\mathcal{E}_{\mathrm{red}})_{\bar 0}\oplus(\mathcal{E}_{\mathrm{red}})_{\bar 1}$, by two locally free sheaves of $\mathcal{F}$-modules

$$(\mathcal{E}_{\mathrm{red}})_{\bar 0}:=\mathcal{E}_{\bar 0}/\mathcal{J}\mathcal{E}\cap\mathcal{E}_{\bar 0}\ \text{ and }\ (\mathcal{E}_{\mathrm{red}})_{\bar 1}:=\mathcal{E}_{\bar 1}/\mathcal{J}\mathcal{E}\cap\mathcal{E}_{\bar 1}$$

of ranks p and q, respectively. Further, the sheaf $\mathcal{E}$ possesses the filtration

$$\mathcal{E}=\mathcal{E}_{(0)}\supset\mathcal{E}_{(1)}\supset\mathcal{E}_{(2)}\supset\cdots,\ \text{where } \mathcal{E}_{(p)}=\mathcal{J}^p\mathcal{E}_{\bar 0}+\mathcal{J}^{p-1}\mathcal{E}_{\bar 1},\ p\geq 1.\tag{2}$$

Using this filtration, we can construct the following locally free sheaf of $\tilde{\mathcal{O}}$-modules on M:

$$\tilde{\mathcal{E}}=\bigoplus\nolimits_p\tilde{\mathcal{E}}_p,\ \text{ where }\ \tilde{\mathcal{E}}_p=\mathcal{E}_{(p)}/\mathcal{E}_{(p+1)}.$$

The sheaf $\tilde{\mathcal{E}}$ is also a locally free sheaf of $\mathcal{F}$-modules. In other words, $\tilde{\mathcal{E}}$ is a sheaf of sections of a certain vector bundle. The following exact sequence gives a description of $\tilde{\mathcal{E}}$.

$$0\to\tilde{\mathcal{O}}_p\otimes(\mathcal{E}_{\mathrm{red}})_{\bar 0}\to\tilde{\mathcal{E}}_p\to\tilde{\mathcal{O}}_{p-1}\otimes(\mathcal{E}_{\mathrm{red}})_{\bar 1}\to 0.$$

We also have the following two exact sequences, which are locally split:

$$\begin{array}{c}0\to\mathcal{E}_{(1)\bar 0}\to\mathcal{E}_{(0)\bar 0}\xrightarrow{\alpha}\tilde{\mathcal{E}}_0\to 0;\\ 0\to\mathcal{E}_{(2)\bar 1}\to\mathcal{E}_{(1)\bar 1}\xrightarrow{\beta}\tilde{\mathcal{E}}_1\to 0.\end{array}\tag{3}$$

The sheaf $\tilde{\mathcal{E}}$ is $\mathbb{Z}$-graded by definition. Unlike the $\mathbb{Z}_2$-grading considered in [1], the natural $\mathbb{Z}_2$-grading is compatible with this $\mathbb{Z}$-grading.

$$(\tilde{\mathcal{E}})_{\bar{0}} := \bigoplus_{p=2k} \tilde{\mathcal{E}}_p, \quad (\tilde{\mathcal{E}})_{\bar{1}} := \bigoplus_{p=2k_1} \tilde{\mathcal{E}}_p.$$

1.2. Classification theorem for locally free sheaves $\mathcal{E}$ on supermanifolds with given $\tilde{\mathcal{E}}$

Our objective now is to classify locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules on supermanifolds $(M, \mathcal{O})$ which have the fixed retract $(M, \tilde{\mathcal{O}})$ and such that the corresponding locally free sheaf $\tilde{\mathcal{E}}$ is fixed.

Let $(M, \mathcal{O})$ and $(M, \mathcal{O}')$ be two supermanifolds, $\mathcal{E}$, $\mathcal{E}'$ be locally free sheaves of $\mathcal{O}$-modules and $\mathcal{O}'$-modules on M, respectively. Suppose that $\Psi : \mathcal{O} \to \mathcal{O}'$ is a superalgebra sheaf morphism. A vector space sheaf morphism $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is called a *quasi-morphism* if

$$\Phi_\Psi(fv) = \Psi(f)\Phi_\Psi(v), \ f \in \mathcal{O}, \ v \in \mathcal{E}.$$

As usual, we assume that $\Phi_\Psi(\mathcal{E}_{\bar{i}}) \subset \mathcal{E}'_{\bar{i}}$, $\bar{i} \in \{\bar{0}, \bar{1}\}$. An invertible quasi-morphism is called a *quasi-isomorphism*. A quasi-isomorphism $\Phi_\Psi : \mathcal{E} \to \mathcal{E}$ is also called a *quasi-automorphism* of $\mathcal{E}$. Denote by $\mathcal{A}ut\mathcal{E}$ the sheaf of quasi-automorphisms of $\mathcal{E}$. It has a double filtration by the subsheaves

$$\mathcal{A}ut_{(p)(q)}\mathcal{E} := \{\Phi_\Psi \in \mathcal{A}ut\mathcal{E} \mid \Phi_\Psi(v) \equiv v \bmod \mathcal{E}_{(p)}, \ \Psi(f) = f \bmod \mathcal{J}^q \\ \text{for } v \in \mathcal{E}, f \in \mathcal{O}\}, \ p, q \geq 0.$$

We also define the following subsheaf of $\mathcal{A}ut\tilde{\mathcal{E}}$:

$$\widetilde{\mathcal{A}ut}\tilde{\mathcal{E}} := \{\Phi_\Psi \mid \Phi_\Psi \in \mathcal{A}ut(\tilde{\mathcal{E}}), \ \Phi_\Psi \text{ preserves the } \mathbb{Z}\text{-grading of } \tilde{\mathcal{E}}\}. \qquad (4)$$

If $\Phi_\Psi \in \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}}$, then $\Psi : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ also preserves the $\mathbb{Z}$-grading. The 0th cohomology group $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ acts on the sheaf $\mathcal{A}ut\tilde{\mathcal{E}}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ and $\delta \in \mathcal{A}ut\tilde{\mathcal{E}}$. It is easy to see that this action leaves invariant the subsheaves $\mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}}$ and hence induces an action of $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ on the cohomology set $H^1(M, \mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}})$. The unit element $\epsilon \in H^1(M, \mathcal{A}ut_{(p)(q)}\mathcal{E}')$ is a fixed point with respect to the action of $H^0(M, \mathcal{A}ut\mathcal{E}')$.

Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote

$$[\mathcal{E}] = \{\mathcal{E}' \mid \mathcal{E}' \text{ is quasi-isomorphic to } \mathcal{E}\}.$$

The total space of the bundle corresponding to a locally free sheaf $\mathcal{E}$ will be denoted by $\mathbb{E}$. It is a supermanifold. The locally free sheaf $\tilde{\mathcal{E}}$ corresponding to $\mathcal{E}$ has the following property: The retract $\tilde{\mathbb{E}}$ of $\mathbb{E}$ is the total space of the bundle corresponding to $\tilde{\mathcal{E}}$.

Theorem 1.1. *Let $(M, \mathcal{O}')$ be a split supermanifold and $\mathcal{E}'$ be a locally free sheaf of $\mathcal{O}'$-modules on M such that $\mathcal{E}' \simeq \tilde{\mathcal{E}}'$. Then*

$$\{[\mathcal{E}] \mid \tilde{\mathcal{O}} = \mathcal{O}', \ \tilde{\mathcal{E}} = \mathcal{E}'\} \stackrel{1:1}{\longleftrightarrow} H^1(M, \mathcal{A}ut_{(2)(2)}\mathcal{E}')/H^0(M, \widetilde{\mathcal{A}ut}\mathcal{E}').$$

The orbit of the unit element ϵ, which is ϵ itself, corresponds to $\mathcal{E}'$.

Proof. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on $(M,\mathcal{O})$ and $\mathcal{U}=\{U_i\}$ be an open covering of M such that (1) and (3) are split over U_i and $\mathcal{E}|_{U_i}$ are free. In this case, $\tilde{\mathcal{E}}|_{U_i}$ are free sheaves of $\tilde{\mathcal{O}}$-modules. We fix homogeneous bases (even and odd, respectively) $(\hat{e}^i_j)$ and $(\hat{f}^i_j)$ of the free sheaves of $\tilde{\mathcal{O}}$-modules $\tilde{\mathcal{E}}|_{U_i}$, $U_i\in\mathcal{U}$. Without loss of generality, we may assume that $\hat{e}^i_j\in\tilde{\mathcal{E}}_0$ and $\hat{f}^i_j\in\tilde{\mathcal{E}}_1$. We are going to define an isomorphism $\delta_i:\mathcal{E}|_{U_i}\to\tilde{\mathcal{E}}|_{U_i}$.

Let $e^i_j\in\mathcal{E}_{(0)\bar{0}}$ be such that $\alpha(e^i_j)=\hat{e}^i_j$ and $f^i_j\in\mathcal{E}_{(0)\bar{1}}$ be such that $\beta(f^i_j)=\hat{f}^i_j$, see (3). Then (e^i_j,f^i_j) is a local basis of $\mathcal{E}|_{U_i}$. A splitting of (1) determines a local isomorphism $\sigma_i:\mathcal{O}|_{U_i}\to\tilde{\mathcal{O}}|_{U_i}$, see [3]. We put

$$\delta_i\Big(\sum h_je^i_j+\sum g_jf^i_j\Big)=\sum\sigma_i(h_j)\hat{e}^i_j+\sum\sigma_i(g_j)\hat{f}^i_j,\ \ h_j,g_j\in\mathcal{O}.$$

Obviously, δ_i is an isomorphism. We put $\gamma_{ij}:=\sigma_i\circ\sigma_j^{-1}$ and $(g_{ij})_{\gamma_{ij}}:=\delta_i\circ\delta_j^{-1}$. Moreover, $(\gamma_{ij})\in Z^1(\mathcal{U},\mathcal{A}ut_{(2)}\tilde{\mathcal{O}})$, see [3] for more details. We want to show that

$$((g_{ij})_{\gamma_{ij}})\in Z^1(\mathcal{U},\mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}}).$$

Let us take $v\in\tilde{\mathcal{E}}|_{U_j}$, $v=\sum h_k\hat{e}^i_k+\sum g_k\hat{f}^i_k,\ \ h_j,g_j\in\tilde{\mathcal{O}}$. Then by definition we have

$$\delta_j^{-1}(v)=\sum\sigma_j^{-1}(h_k)e^j_k+\sum\sigma_j^{-1}(g_k)f^j_k.$$

The transition functions of $\tilde{\mathcal{E}}$ may be expressed in $U_i\cap U_j$ as follows:

$$e^j_k=\sum a^k_se^i_s+\sum b^k_sf^i_s,\ f^j_k=\sum c^k_se^i_s+\sum d^k_sf^i_s,\ a^k_s,d^k_s\in\mathcal{O}_{\bar{0}},\ b^k_s,c^k_s\in\mathcal{O}_{\bar{1}}.$$

Further,

$$\alpha(e^j_k)=\hat{e}^j_k=\sum\pi(a^k_s)\hat{e}^i_s,\quad\beta(f^j_k)=\hat{f}^j_k=\sum\tau(c^k_s)\hat{e}^i_s+\sum\pi(d^k_s)\hat{f}^i_s.$$

We have

$$\begin{aligned}\delta_j\circ\delta_j^{-1}(v)=&\sum_k\gamma_{ij}(h_k)\Big(\sum_s\sigma_i(a^k_s)\hat{e}^i_s+\sum_r\sigma_i(b^k_s)\hat{f}^i_s\Big)\\&+\sum_k\gamma_{ij}(g_k)\Big(\sum_s\sigma_i(c^k_s)\hat{e}^i_s+\sum_s\sigma_i(d^k_s)\hat{f}^i_s\Big)\\=&\sum_kh_k\Big(\sum_s\pi(a^k_s)\hat{e}^i_s\Big)\\&+\sum_kg_k\Big(\sum_s\tau(c^k_s)\hat{e}^i_s+\sum_s\pi(d^k_s)\hat{f}^i_s\Big)\operatorname{mod}\tilde{\mathcal{E}}_{(2)}\\=&\,v\operatorname{mod}\tilde{\mathcal{E}}_{(2)}.\end{aligned}$$

The rest of the proof is the direct repetition of the proof of Theorem 2 from [1]. □

2. The spectral sequence

2.1. Quasi-derivations

Quasi-derivations were defined in [1]. Let us briefly recall that construction. Consider a locally free sheaf $\mathcal{E}$ on a supermanifold $(M,\mathcal{O})$. An even vector space sheaf morphism $A_\Gamma : \mathcal{E} \to \mathcal{E}$ is called a *quasi-derivation* if $A_\Gamma(fv) = \Gamma(f)v + fA_\Gamma(v)$, where $f \in \mathcal{O}$, $v \in \mathcal{E}$ and Γ is a certain even super vector field. Denote by $\operatorname{Der}\mathcal{E}$ the sheaf of quasi-derivations. It is a sheaf of Lie algebras with respect to the commutator $[A_\Gamma, B_\Upsilon] := A_\Gamma \circ B_\Upsilon - B_\Upsilon \circ A_\Gamma$. The sheaf $\operatorname{Der}\mathcal{E}$ possesses a double filtration

$$\begin{array}{ccccc} \operatorname{Der}_{(0)(0)}\mathcal{E} & \supset & \operatorname{Der}_{(2)(0)}\mathcal{E} & \supset \cdots \\ \cup & & \cup & \\ \operatorname{Der}_{(0)(2)}\mathcal{E} & \supset & \operatorname{Der}_{(2)(2)}\mathcal{E} & \supset \cdots, \\ \vdots & & \vdots & \end{array}$$

where

$$\operatorname{Der}_{(p)(q)}\mathcal{E} := \{A_\Gamma \in \operatorname{Der}\mathcal{E} \mid A_\Gamma(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)},\ \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q},\ r, s \in \mathbb{Z}\},$$

where $p, q \geq 0$. The map defined by the usual exponential series

$$\exp : \operatorname{Der}_{(p)(q)}\mathcal{E} \to \mathcal{A}ut_{(p)(q)}\mathcal{E},\ p, q \geq 2,$$

is an isomorphism of sheaves of sets, because operators from $\operatorname{Der}_{(p)(q)}\mathcal{E}$, $p, q \geq 2$, are nilpotent. The inverse map is given by the logarithmic series. Define the vector space subsheaf $\operatorname{Der}_{k,k}\tilde{\mathcal{E}}$ of $\operatorname{Der}_{(k)(k)}\tilde{\mathcal{E}}$ for $k \geq 0$ by

$$\operatorname{Der}_{k,k}\tilde{\mathcal{E}} := \{A_\Gamma \in \operatorname{Der}_{(k)(k)}\tilde{\mathcal{E}} \mid A_\Gamma(\tilde{\mathcal{E}}_r) \subset \tilde{\mathcal{E}}_{r+k},\ \Gamma(\tilde{\mathcal{O}}_s) \subset \tilde{\mathcal{O}}_{s+k},\ r, s \in \mathbb{Z}\}.$$

For an even $k \geq 2$, define a map

$$\mu_k : \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}} \to \operatorname{Der}_{k,k}\tilde{\mathcal{E}}, \quad \mu_k(a_\gamma) = \bigoplus_q \operatorname{pr}_{q+k} \circ A_\Gamma \circ \operatorname{pr}_q,$$

where $a_\gamma = \exp(A_\Gamma)$ and $\operatorname{pr}_k : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}_k$ is the natural projection. The kernel of this map is $\mathcal{A}ut_{(k+2)(2)}\tilde{\mathcal{E}}$. Moreover, the sequence

$$0 \to \mathcal{A}ut_{(k+2)(2)}\tilde{\mathcal{E}} \longrightarrow \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}} \xrightarrow{\mu_k} \operatorname{Der}_{k,k}\tilde{\mathcal{E}} \to 0,$$

where $k \geq 2$ is even, is exact. Denoting by $H_{(k)}(\tilde{\mathcal{E}})$ the image of the natural mapping $H^1(M, \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}}) \to H^1(M, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}})$, we get the filtration

$$H^1(M, \mathcal{A}ut_{(2)}\tilde{\mathcal{E}}) = H_{(2)}(\tilde{\mathcal{E}}) \supset H_{(4)}(\tilde{\mathcal{E}}) \supset \cdots.$$

Take $a_\gamma \in H_{(2)}(\tilde{\mathcal{E}})$. We define the *order of* a_γ to be the maximal number k such that $a_\gamma \in H_{(k)}(\tilde{\mathcal{E}})$. The *order of a locally free sheaf* $\mathcal{E}$ of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O}_M)$ is by definition the order of the corresponding cohomology class.

2.2. The spectral sequence

A spectral sequence connecting the cohomology with values in the tangent sheaf $\mathcal{T}$ of a supermanifold $(M, \mathcal{O})$ with the cohomology with values in the tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ of the retract $(M, \tilde{\mathcal{O}})$ was constructed in [2]. Here we use similar ideas to construct a new spectral sequence connecting the cohomology with values in a locally free sheaf $\mathcal{E}$ on a supermanifold $(M, \mathcal{O})$ with the cohomology with values in the locally free sheaf $\tilde{\mathcal{E}}$ on $(M, \tilde{\mathcal{O}})$. Note that our spectral sequence is not a generalization of the spectral sequence obtained in [2] because $\mathcal{T}_{\mathrm{gr}}$ is not in general isomorphic to $\tilde{\mathcal{T}}$.

Let $\mathcal{E}$ be a locally free sheaf on a supermanifold $(M,\mathcal{O})$ of dimension $n|m$. We fix an open Stein covering $\mathfrak{U}=(U_i)_{i\in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathfrak{U},\mathcal{E})=\bigoplus_{p\geq 0}C^p(\mathfrak{U},\mathcal{E})$. The $\mathbb{Z}_2$-grading of $\mathcal{E}$ gives rise to the $\mathbb{Z}_2$-gradings in $C^*(\mathfrak{U},\mathcal{E})$ and $H^*(M,\mathcal{E})$ given by

$$\begin{aligned} C_{\bar 0}(\mathfrak{U},\mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U},\mathcal{E}_{\bar 0})\oplus\bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U},\mathcal{E}_{\bar 1}),\\ C_{\bar 1}(\mathfrak{U},\mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U},\mathcal{E}_{\bar 1})\oplus\bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U},\mathcal{E}_{\bar 0}).\\ H_{\bar 0}(M,\mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M,\mathcal{E}_{\bar 0})\oplus\bigoplus_{q\geq 0} H^{2q+1}(M,\mathcal{E}_{\bar 1}),\\ H_{\bar 1}(M,\mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M,\mathcal{E}_{\bar 1})\oplus\bigoplus_{q\geq 0} H^{2q+1}(M,\mathcal{E}_{\bar 0}). \end{aligned} \tag{5}$$

The filtration (2) for $\mathcal{E}$ gives rise to the filtration

$$C^*(\mathfrak{U},\mathcal{E})=C_{(0)}\supset\cdots\supset C_{(p)}\supset\cdots\supset C_{(m+2)}=0 \tag{6}$$

of this complex by the subcomplexes

$$C_{(p)}=C^*(\mathfrak{U},\mathcal{E}_{(p)}).$$

Denoting by $H(M,\mathcal{E})_{(p)}$ the image of the natural mapping $H^*(M,\mathcal{E}_{(p)})\to H^*(M,\mathcal{E})$, we get the filtration

$$H^*(M,\mathcal{E})=H(M,\mathcal{E})_{(0)}\supset\cdots\supset H(M,\mathcal{E})_{(p)}\supset\cdots. \tag{7}$$

Denote by $\operatorname{gr}H^*(M,\mathcal{E})$ the bigraded group associated with the filtration (7); its bigrading is given by

$$\operatorname{gr}H^*(M,\mathcal{E})=\bigoplus_{p,q\geq 0}\operatorname{gr}_p H^q(M,\mathcal{E}).$$

By the (more general) Leray procedure, we get a spectral sequence of bigraded groups E_r converging to $E_\infty\simeq\operatorname{gr}H^*(M,\mathcal{E})$. For convenience of the reader, we recall the main definitions here.

For any $p,r\geq 0$, define the vector spaces

$$C^p_r=\{c\in C_{(p)}\,|\,dc\in C_{(p+r)}\}.$$

Then, for a fixed p, we have

$$C_{(p)}=C^p_0\supset\cdots\supset C^p_r\supset C^p_{r+1}\supset\cdots.$$

The rth term of the spectral sequence is defined by

$$E_r=\bigoplus_{p=0}^{m}E^p_r,\ r\geq 0,\ \text{where } E^p_r=C^p_r/C^{p+1}_{r-1}+dC^{p-r+1}_{r-1}.$$

Since $d(C_r^p) \subset C_r^{p+r}$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. The $\mathbb{Z}_2$-grading (5) in $C^*(\mathfrak{U}, \mathcal{E})$ gives rise to certain $\mathbb{Z}_2$-gradings in C_r^p and E_r^p, turning E_r into a superspace. Clearly, the coboundary operator d on $C^*(\mathfrak{U}, \mathcal{E})$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superspaces E_r are also endowed with a second $\mathbb{Z}$-grading. Namely, for any $q \in \mathbb{Z}$, set

$$C_r^{p,q} = C_r^p \cap C^{p+q}(\mathfrak{U}, \mathcal{E}), \quad E_r^{p,q} = C_r^{p,q}/C_{r-1}^{p+1,q-1} + dC_{r-1}^{p-r+1,q+r-2}.$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q} \text{ and } d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1} \text{ for any } r,\, p,\, q. \tag{8}$$

Further, for a fixed q, we have $d(C_r^{p,q}) = 0$ for all $p \geq 0$ and all $r \geq m + 2$. This implies that the natural homomorphism $E_r^{p,q} \to E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0 = m + 2$. Setting $E_\infty^{p,q} = E_{r_0}^{p,q}$, we get the bigraded superspace

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Lemma 2.1. *The first two terms of the spectral sequence* (E_r) *can be identified with the following bigraded spaces:*

$$E_0 = C^*(\mathfrak{U}, \tilde{\mathcal{E}}),\ E_1 = E_2 = H^*(M, \tilde{\mathcal{E}}).$$

More precisely,

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, \tilde{\mathcal{E}}_p),\ E_1^{p,q} = E_2^{p,q} = H^{p+q}(M, \tilde{\mathcal{E}}_p).$$

We have $d_{2k+1} = 0$ *and, hence,* $E_{2k+1} = E_{2k+2}$ *for all* $k \geq 0$.

Proof. The proof is similar to the proof of Proposition 3 in [2]. □

Lemma 2.2. *There is the following identification of bigraded algebras:*

$$E_\infty = \operatorname{gr} H^*(M, \mathcal{E}),\ \textit{where } E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).$$

If M *is compact, then* $\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E_\infty^{p,q}$.

Proof. The proof is a direct repetition of the proof of Proposition 4 in [2]. □

Now we prove our main result concerning the first non-zero coboundary operators among $d_2,\ d_4, \ldots$. Assume that the isomorphisms of sheaves $\delta_i : \mathcal{E}|U_i \to \tilde{\mathcal{E}}|U_i$ from Theorem 1.1 are defined for each $i \in I$. By Theorem 1.1, a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on M corresponds to the cohomology class a_γ of the 1-cocycle $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}})$, where $(a_\gamma)_{ij} = \delta_i \circ \delta_j^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to k, then we may choose $\delta_i,\ i \in I$, in such a way that $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}})$. We can write $a_\gamma = \exp A_\Gamma$, where $A_\Gamma \in C^1(\mathfrak{U}, \operatorname{Der}_{(k)(2)} \tilde{\mathcal{E}})$.

We will identify the superspaces (E_0, d_0) and $(C^*(\mathfrak{U}, \tilde{\mathcal{E}}), d)$ via the isomorphism of Lemma 2.1. Clearly, $\delta_i : \mathcal{E}_{(p)}|U_i \to \tilde{\mathcal{E}}_{(p)}|U_i = \sum_{r\geq p} \tilde{\mathcal{E}}_r|U_i$ is an isomorphism of sheaves for all $i \in I,\ p \geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi : C^*(\mathfrak{U}, \mathcal{E}) \to C^*(\mathfrak{U}, \tilde{\mathcal{E}})$$

such that

$$\psi : C^*(\mathfrak{U}, \mathcal{E}_{(p)}) \to C^*(\mathfrak{U}, (\tilde{\mathcal{E}})_{(p)}),\ p \geq 0.$$

We put

$$\psi(c)_{i_0 \ldots i_q} = \delta_{i_0}(c_{i_0 \ldots i_q})$$

for any $(i_0, \ldots, i_q)$ such that $U_{i_0} \cap \cdots \cap U_{i_q} \neq \emptyset$. Note that ψ is not an isomorphism of complexes. Nevertheless, we can explicitly express the coboundary d of the complex $C^*(\mathfrak{U}, \mathcal{E})$ by means of d_0 and a_γ.

The following theorem permits to calculate the spectral sequence (E_r) whenever d_0 and the cochain a_γ are known. It also describes certain coboundary operators $d_r,\ r \geq 1$.

Theorem 2.3. *For any $c \in C^*(\mathfrak{U}, \tilde{\mathcal{E}}_q) = E_0^q$, we have*

$$(\psi(d\psi^{-1}(c)))_{i_0 \ldots i_{q+1}} = (d_0 c)_{i_0 \ldots i_{q+1}} + ((a_\gamma)_{i_0 i_1} - \mathrm{id})(c_{i_1 \ldots i_{q+1}}).$$

Suppose that the locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on M has order k and denote by a_γ the cohomology class corresponding to $\mathcal{E}$ by Theorem 1.1*. Then $d_r = 0$ for $r = 1, \ldots, k-1$, and $d_k = \mu_k(a_\gamma)$.*

Proof. The proof is similar to the proof of Theorem 7 in [1]. □

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Bogdan Mielnik: Contributions to Quantum Control

David J. Fernández C.

To Professor Bogdan Mielnik on his 75th Birthday

Abstract. In this article two main aspects of quantum control, which require basically different mathematical techniques will be addressed. In the first one the systems are characterized by stationary Hamiltonians, while in the second they are ruled by time-dependent ones. Both trends were initiated in Mexico by Bogdan Mielnik, who has played a central role in the development of a research group on quantum control at Cinvestav.

Mathematics Subject Classification (2010). Primary 81Q05; Secondary 81Q60.

Keywords. Quantum control, Factorization method, Supersymmetric quantum mechanics.

1. Introduction

I would like to describe here the genesis and development of the quantum control group created by Bogdan Mielnik (BM) at the Center for Advanced Studies (Cinvestav) in Mexico City. Indeed, the beginning of this story is strongly tied to the birth of our Physics Department at Cinvestav, which deserves some words.

In 1962, while working at the Institute of Theoretical Physics of Warsaw University, Jerzy Plebański was invited by the outstanding Mexican physiologist Arturo Rosenblueth to develop a Physics Department at the recently created Cinvestav, at the north of Mexico City. In that invitation, it was suggested that Jerzy should also invite a younger assistant from Poland, to help him do the job. Plebański accepted Rosenblueth's invitation, and he arrived to Mexico in the late summer of 1962. His younger fellow, who turned out to be Bogdan Mielnik, arrived to Mexico on November 13th, 1962 as Jerzy's assistant and his Ph.D. student. From this period (1963) is the photograph in which Jerzy Plebański, Bogdan Mielnik and Anna Plebańska stay in front of the Pyramid of Quetzalcóatl, at the Teotihuacán ceremonial center (see Figure 1).

FIGURE 1. Jerzy Plebański, Bogdan Mielnik and Anna Plebańska in front of the Pyramid of Quetzalcóatl, at the Teotihuacán ceremonial center (1963).

During his first stay at Mexico, Mielnik taught courses on the mathematical foundations of quantum mechanics. As for research, he was working on the finite difference calculus and pseudo-hermitian operators. On October 22nd, 1964, he submitted his PhD Thesis entitled *Analytic functions of the displacement operator* [1] (see also [2]). Incidentally, it is worth mentioning that Bogdan Mielnik was the first Ph.D. graduate of our Physics Department at Cinvestav. A copy of the official document is shown in Figure 2.

In April 1965, after finishing his PhD, Mielnik returned to Poland. In the following years, he maintained interest in the operator calculus, leading to the explicit algebraic solution of the continuous Baker-Campbell-Hausdorff (BCH) problem [3,4], which remained open for about 60 years since the original BCH papers. In the period 1966–1976, Mielnik wrote and published his seminal papers on the geometric structure of quantum theories [5–8]. Due to the wide impact of these works, he was invited, in the period 1975–1980, to several prestigious institutions, both in Europe and in the United States, such as the Institute of Theoretical Physics in Gothenburg and the Royal Institute of Technology in Stockholm (Sweden), King's College and Imperial College (United Kingdom), Rockefeller University (USA), among others (see, e.g., [9,10]). In particular, in 1976 and 1978 he got back to Mexico to deliver talks at the *International Symposium on Mathematical Physics* in the old *Hotel del Prado*, destroyed by the earthquake in 1985, and the *Latin American Symposium on General Relativity* (Silarg) [11,12]. From that time

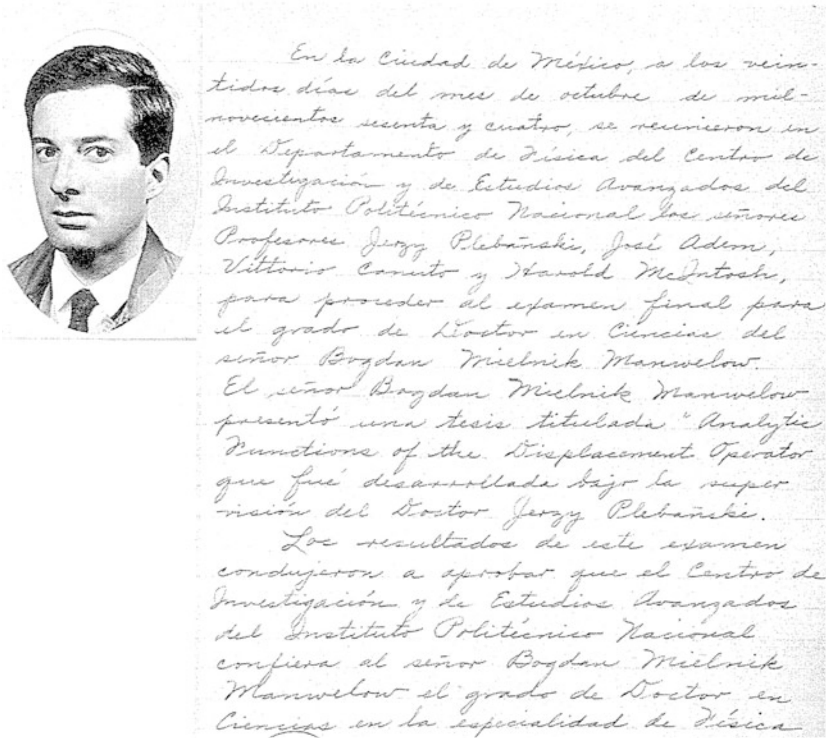
En la Ciudad de México, a los veintidos días del mes de octubre de milnovecientos sesenta y cuatro, se reunieron en el Departamento de Física del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional los señores Profesores Jerzy Plebański, José Adem, Vittorio Canuto y Harold McIntosh, para proceder al examen final para el grado de Doctor en Ciencias del señor Bogdan Mielnik Manwelow.
El señor Bogdan Mielnik Manwelow presentó una tesis titulada "Analytic Functions of the Displacement Operator" que fué desarrollada bajo la supervisión del Doctor Jerzy Plebański.
Los resultados de este examen condujeron a aprobar que el Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional confiera al señor Bogdan Mielnik Manwelow el grado de Doctor en Ciencias en la especialidad de Física

FIGURE 2. The copy of the Mielnik's Ph.D. certificate (October 22nd, 1964) from the official Cinvestav roster (from S. Quintanilla, *Recordar hacia el mañana. Creación y primeros años del Cinvestav* 1960–1970, Cinvestav, Mexico, 2002).

(1976) is the nice photograph in which Bogdan Mielnik, Anna Plebańska, Virginia T. Rosenblueth and Pleblański's daughter Magdalena appear in front of some already non-existent buildings in the Reforma Avenue in Mexico City (see Figure 3).

In November 1981, Mielnik visited Cinvestav, in what was supposed to be a short-term visit. This seemingly current event became crucial for our Department and for Mielnik's life. In December 13th, 1981, while he was still in Mexico, the martial law was declared in Poland. The situation seemed to be hard in Warsaw and thus Augusto García, at the time Head of the Department, proposed Mielnik to stay longer at Cinvestav. He decided to accept this invitation which, as the years passed by, turned into a permanent stay. During that time Mielnik pursued his studies on dynamical manipulation [13], and he also wrote his short seminal article, about the generation of new Hamiltonians isospectral to the harmonic oscillator through a variant of the factorization method [14]. In the early 1983 I met Bogdan Mielnik as a student of his course in quantum mechanics. I was subsequently

FIGURE 3. Bogdan Mielnik, Anna Plebańska, Virginia T. Rosenblueth and Magdalena Plebański in *Paseo de la Reforma*, Mexico City (January 1976).

involved, already as Mielnik's MSc student, in applying the recently developed modified factorization to the Coulomb potential. The photograph of Mielnik in his office in *Física II* (see Figure 4) is from that time (1986). In the following years 1986–1987, he spent a sabbatical leave at the Institute of Theoretical Physics of Warsaw University.

In the period 1987–1990, Mielnik got a double appointment at the Physics Department of Cinvestav and the Institute of Theoretical Physics of Warsaw University. In 1989 he was nominated the Full Professor at the Institute of Theoretical Physics of Warsaw University. In parallel he has been a Permanent Professor at Cinvestav.

During all the time that he spent in Mexico, Mielnik has produced outstanding works, becoming the founder of the quantum control school currently existing at Cinvestav. The motivation of this subject is to control typically quantum phenomena such as diffraction, interference, wave-packet spreading, decoherence, etc. Our dream is to build a handbook of unitary operations that can be dynamically achieved.

On the other hand, for stationary systems the equivalent goal would be to construct Hamiltonians with an *a priori* prescribed spectrum. The first steps in that direction have been given by employing the well-known *factorization method*, which is worth describing shortly.

FIGURE 4. Bogdan Mielnik at his office in *Física II*, Cinvestav (1986).

2. Control of systems with time-independent Hamiltonians

When dealing with stationary systems, an obvious target to manipulate is the Hamiltonian spectrum. The simplest available technique for spectral manipulation is the factorization method, which is equivalent to the intertwining technique, Darboux transformation and supersymmetric quantum mechanics. The way in which the factorization method works can be simply illustrated through the harmonic oscillator potential.

The harmonic oscillator Hamiltonian in natural units, with $\hbar = m = \omega = 1$, reads

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2}. \tag{1}$$

The standard factorizations in terms of the annihilation a and creation a^+ operators are given by:

$$H = aa^+ - \frac{1}{2}, \tag{2}$$

$$H = a^+a + \frac{1}{2}, \tag{3}$$

where

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \tag{4}$$

$$a^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right). \tag{5}$$

From these expressions the following intertwining relationships can be derived:

$$Ha^+ = a^+(H+1), \tag{6}$$

$$Ha = a(H-1), \tag{7}$$

which imply that, by acting the operator a (a^+) onto an eigenfunction of H with eigenvalue E, a new eigenfunction of H is obtained with eigenvalue $E-1$ ($E+1$). By using all these ingredients, it is straightforward to derive the complete set of eigenfunctions $\psi_n(x)$ and eigenvalues $E_n = n + 1/2$ of H, for $n = 0, 1, \ldots$

In 1983 Mielnik asked a simple question [14]: Is the factorization of the harmonic oscillator Hamiltonian given in equation (2) unique? In order to answer, he looked for more general first-order differential operators

$$b = \frac{1}{\sqrt{2}}\left[\frac{d}{dx} + \beta(x)\right], \tag{8}$$

$$b^+ = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + \beta(x)\right], \tag{9}$$

such that

$$H = bb^+ - \frac{1}{2}. \tag{10}$$

It turns out that the unknown function $\beta(x)$ must satisfy the Riccati equation

$$\beta' + \beta^2 = x^2 + 1, \tag{11}$$

whose general solution is given by

$$\beta = x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2} dy}. \tag{12}$$

The key point now is that the product b^+b, in general, is no longer reduced to the harmonic oscillator Hamiltonian, but it leads to a different operator:

$$\widetilde{H} = b^+b + \frac{1}{2} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x), \tag{13}$$

where

$$\widetilde{V}(x) = \frac{x^2}{2} - \frac{d}{dx}\left[\frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2} dy}\right]. \tag{14}$$

However, there are still intertwining relationships that look similar to those of equations (6) and (7),

$$\widetilde{H}b^+ = b^+(H+1), \tag{15}$$

$$Hb = b(\widetilde{H}-1). \tag{16}$$

Thus, the eigenfunctions $\widetilde{\psi}_n$ of $\widetilde{H}$ can be easily constructed from those of H:

$$\widetilde{\psi}_{n+1} = \frac{b^+\psi_n}{\sqrt{n+1}}, \quad n = 0, 1, \ldots \tag{17}$$

Moreover, there is an additional eigenstate of $\widetilde{H}$ associated to the eigenvalue $E_0 = 1/2$ and simultaneously annihilated by b, which is given by:

$$\widetilde{\psi}_0 \propto \exp\left[-\int_0^x \beta(y)dy\right]. \tag{18}$$

In order to avoid singularities in $\widetilde{V}(x)$ and in $\widetilde{\psi}_n(x),\ n = 0, 1, \ldots$, the inequality $|\lambda| > \sqrt{\pi}/2$ must hold. Thus, in this λ-domain it turns out that $\widetilde{H}$ is a new Hamiltonian isospectral to the harmonic oscillator.

FIGURE 5. Bogdan Mielnik, David Fernández and Oscar Rosas, during a break at the Conference *Symmetries in quantum mechanics and quantum optics*, Burgos (Spain), September of 1998.

It is worth noting that the modified factorization described here represented a breakthrough in the generation of exactly solvable quantum mechanical potentials. Indeed, the intertwining relation (15) admits several generalizations that were proposed shortly after. An obvious one consists in departing from a given generic Schrödinger Hamiltonian H of the form (13) and look for a new one $\widetilde{H}$ such that

$$\widetilde{H}B^+ = B^+H, \tag{19}$$

where the initial potential $V(x)$ and the intertwining operator B^+ are not necessarily the harmonic oscillator and a first-order operator respectively. In particular,

Figure 6. Boris Samsonov, David Fernández, Bogdan Mielnik and Oscar Rosas at Mielnik's office (March of 2001).

the generalization for B^+ being of first-order and general $V(x)$ was proposed by Sukumar in 1985, who proved that a solution of the stationary Schrödinger equation associated to H and a given factorization energy ϵ such that $\epsilon \leq E_0$ is required to generate the new potential $\widetilde{V}(x)$ through non-singular transformations. On the other hand, Andrianov, Ioffe and Spiridonov (1993) suggested that B^+ should be of order greater than one with general $V(x)$, and this suggestion was later studied by Andrianov, Ioffe, Cannata, Dedonder (1995), Bagrov and Samsonov (1995) and a member of our research group (Fernández 1997). It is important to notice that in the higher-order case several seed solutions of the stationary Schrödinger equation associated to diverse factorization energies are required in order to implement the transformation (for a review containing further discussion, the reader can consult [15]).

The case where $V(x)$ is the harmonic oscillator potential and B^+ is of second-order was explored in detail in 1998 by members of our group [16]. A photograph taken during a break at the Conference *Symmetries in quantum mechanics and quantum optics* which was held at Burgos, Spain, can be seen on Figure 5. Subsequently, the so-called confluent algorithm, for which the involved factorization energies tend to a common value, was explored in 2000 by Mielnik, Nieto and Rosas-Ortiz [17], and later by Fernández and Salinas-Hernández. The situation when $V(x)$ is periodic has been also analyzed in the interval 2000–2010 (see, e.g., [18, 19]). Some members of our team elaborating the last problem appear on the photo of Figure 6.

Before finishing this section, I would like to remark that in 2003 the Conference *Progress in supersymmetric quantum mechanics* took place at Valladolid, Spain. An overview article opening the special issue of *J. Phys. A: Math. Gen.* dedicated to the topic of the Conference, that has quickly became a hit of the factorization subject, is strongly recommended (see [15]).

3. Control of systems with time-dependent Hamiltonians

For systems ruled by time-dependent Hamiltonians the quantum control has to be implemented in a different way. First of all, it is well known that the evolution operator induced by a self-adjoint Hamiltonian is unitary. Thus, it is natural to consider the inverse problem: Can any unitary operator be achieved as the result of a dynamical evolution? In other words, can a set of prescribed external conditions be designed for the system to evolve in such a way that its evolution operator becomes, at a certain time, the required unitary operator? The answer to this question was suggested by Mielnik in 1977 [20]: provided there are no superselection rules, any unitary operator can be dynamically approximated. Moreover, there is a generic prescription, proposed in 1986, in order to induce an arbitrary unitary evolution [21, 22]: (i) first of all, let us choose the system that performs a circular dynamical process such that $U_0(\tau) = I$, that is called an evolution loop (EL); (ii) then, by perturbing the EL, the small deviations of this process will eventually induce any given unitary operation (see an illustration of this process in Figure 7).

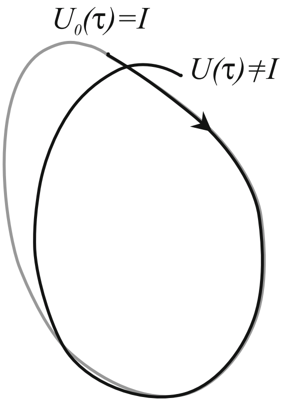

FIGURE 7. The deviation of the evolution loop induced by a perturbation.

3.1. One-dimensional systems

Let us note that the one-dimensional harmonic oscillator is the simplest system having an EL. Thus, it is natural first to look for EL in one-dimensional systems ruled by Hamiltonians of the form

$$H(t) = \frac{p^2}{2} + g(t)\frac{q^2}{2}, \tag{20}$$

where q, p are the quantum mechanical coordinate and momentum operators such that

$$[q, p] = i, \tag{21}$$

and the evolution operator $U(t)$ of the system satisfies

$$\frac{dU(t)}{dt} = -iH(t)U(t), \quad U(0) = I. \tag{22}$$

A curious and interesting result that was found in 1977 deserves some discussion [20]. For the periodic sequence of pulses such that

$$g(t) = \frac{1}{\lambda}\delta(t-\lambda) \quad \text{for} \quad 0 < t \leq \lambda, \tag{23}$$

periodically extended for $t > \lambda$, the following holds

$$\underbrace{e^{-i\frac{1}{\lambda}\frac{q^2}{2}} e^{-i\lambda\frac{p^2}{2}} \cdots e^{-i\frac{1}{\lambda}\frac{q^2}{2}} e^{-i\lambda\frac{p^2}{2}}}_{12 \text{ factors}} \equiv I, \tag{24}$$

where the equivalence symbol $\equiv$ interrelates any two unitary operators which differ only by a c-number phase factor. This means that the system has an evolution loop of period $\tau = 6\lambda$. A schematic representation of this dynamical process is given in Figure 8. Notice that, as a bonus, it is possible now to invert the natural free evolution:

$$e^{i\lambda\frac{p^2}{2}} \equiv \underbrace{e^{-i\frac{1}{\lambda}\frac{q^2}{2}} e^{-i\lambda\frac{p^2}{2}} \cdots e^{-i\lambda\frac{p^2}{2}} e^{-i\frac{1}{\lambda}\frac{q^2}{2}}}_{11 \text{ factors}}. \tag{25}$$

Let us stress that the evolution loop of equation (24) is not the only one that can be produced through Hamiltonians of the form (20) [22]. In particular, it turns out that

$$\left(e^{-i3\tau\frac{p^2}{2}} e^{-i\frac{1}{\tau}\frac{q^2}{2}}\right)^3 \equiv I, \tag{26}$$

which implies that it is possible once again to invert the natural free evolution:

$$e^{i3\tau\frac{p^2}{2}} \equiv e^{-i\frac{1}{\tau}\frac{q^2}{2}} e^{-i3\tau\frac{p^2}{2}} e^{-i\frac{1}{\tau}\frac{q^2}{2}} e^{-i3\tau\frac{p^2}{2}} e^{-i\frac{1}{\tau}\frac{q^2}{2}}. \tag{27}$$

A representation of the evolution loop of equation (26) is also given in Figure 8 [23].

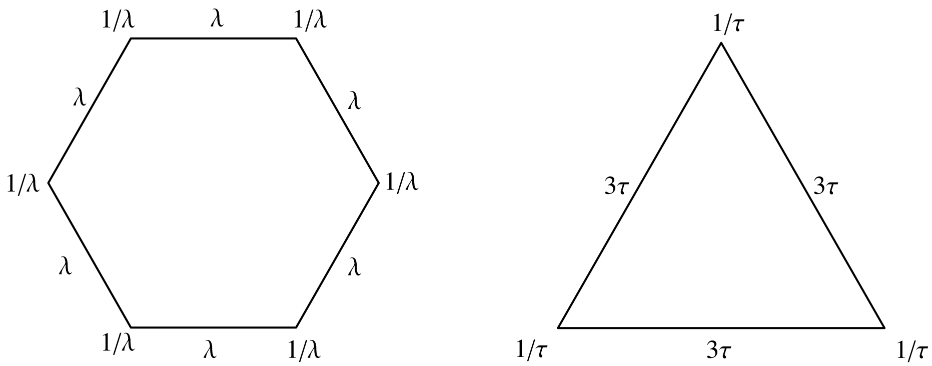

FIGURE 8. Representation of the evolution loops of equations (24) (left) and (26) (right).

3.2. Three-dimensional systems

As our three-dimensional system let us consider a charged particle interacting with homogeneous time-dependent magnetic fields. A possible experimental setup is illustrated in Figure 9. In a neighborhood of the origin, the magnetic field can be considered approximately homogeneous, and the corresponding Hamiltonian takes

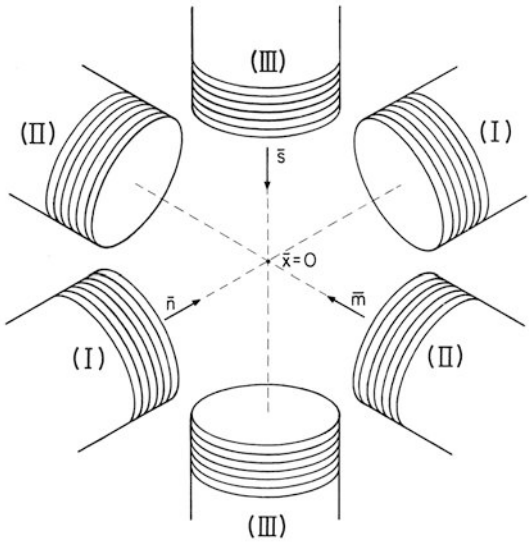

FIGURE 9. An experimental setup to manipulate charged particles.

the form:

$$H(t) = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{2c}\mathbf{r}\times\mathbf{B}(t)\right)^2 = \frac{1}{2m}\left[\mathbf{p}^2 + \left(\frac{e\mathbf{B}(t)}{2c}\right)^2\mathbf{r}_\perp^2\right] - \frac{e\mathbf{B}(t)\cdot\mathbf{L}}{2mc}, \quad (28)$$

a non-relativistic Hamiltonian with time dependent $\mathbf{B}(t)$ representing the first step of the Einstein-Infeld-Hoffman (EIH) method in classical electrodynamics (see the discussion in [24]). Our first choice was the following rotating magnetic field [25]:

$$\mathbf{B}(t) = B\cos(\omega t)\mathbf{m} + B\sin(\omega t)\mathbf{n}, \quad (29)$$

for which we wanted to find the evolution loops. Unfortunately, we were unable to find them for this system. Despite that, the corresponding quantum mechanical problem was explicitly solved. We found a regime where the charged particle is confined to a neighborhood of the origin (the trapping region). However, there exists also the domain of parametric resonance, where the charged particle is quickly ejected off the trapping zone (see also [26–28]). These results constitute the core of my PhD Thesis [29], supervised by Mielnik. The dissertation was delivered on September 19th, 1988 (a photograph of José Luis Lucio, Bogdan Mielnik and David Fernández, after the event, can be seen in Figure 10).

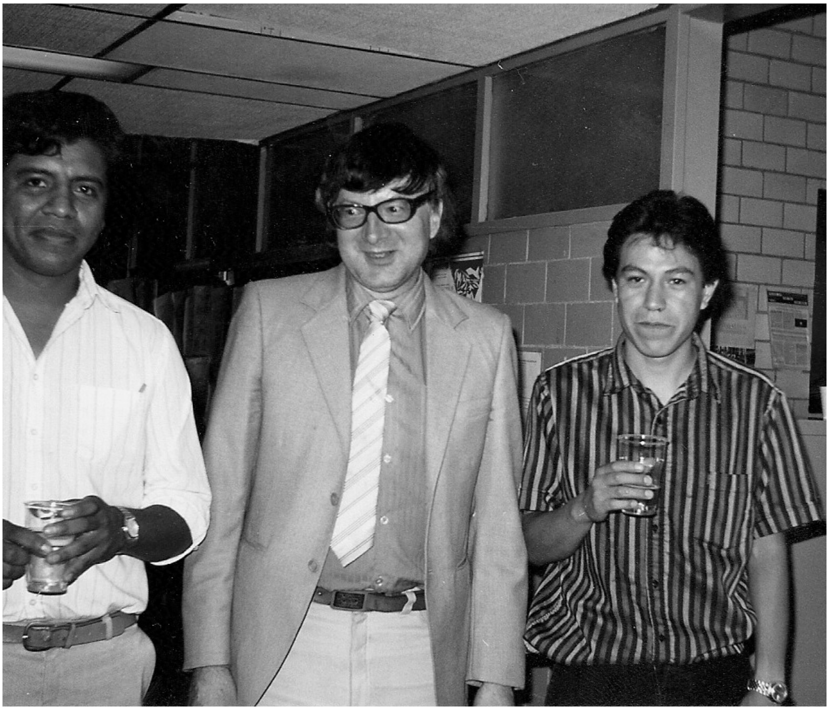

FIGURE 10. José Luis Lucio, Bogdan Mielnik and David Fernández at *Física I*, September 19th, 1988.

An alternative magnetic field was explored afterwards [30] (see also [31–33]):

$$\mathbf{B}(t) = \begin{cases} B(t)\mathbf{m} & \text{for } t \in [0, 2T) \\ B(t-2T)\mathbf{n} & \text{for } t \in [2T, 4T) \\ B(t-4T)\mathbf{s} & \text{for } t \in [4T, 6T) \end{cases}, \tag{30}$$

where

$$B(t) = \begin{cases} B_1 & \text{for } t \in [0, t_1),\ 0 < t_1 < T, \\ B_2 & \text{for } t \in [t_1, T), \\ -B_2 & \text{for } t \in [T, T+t_2),\ t_2 = T - t_1, \\ -B_1 & \text{for } t \in [T+t_2, 2T). \end{cases} \tag{31}$$

For the three-dimensional system we were particularly interested, as in the one-dimensional case, in inverting the natural free evolution. In order to do that, we first of all switched to the following dimensionless quantities:

$$\gamma_1 = \alpha_1 t_1', \quad \gamma_2 = \alpha_2 t_2', \quad \alpha_1 = \frac{eB_1T}{2mc}, \tag{32}$$

$$\alpha_2 = \frac{eB_2T}{2mc}, \quad t_1' = \frac{t_1}{T}, \quad t_2' = \frac{t_2}{T}. \tag{33}$$

It turns out that the free evolution is induced when the previous parameters satisfy the following relationships:

$$\alpha_1 = \frac{\gamma_1 \tan(\gamma_1) - \gamma_2 \tan(\gamma_2)}{\tan(\gamma_1)}, \qquad \alpha_2 = \frac{\gamma_1 \tan(\gamma_1) - \gamma_2 \tan(\gamma_2)}{-\tan(\gamma_2)}, \tag{34}$$

$$t_1' = \frac{\gamma_1 \tan(\gamma_1)}{\gamma_1 \tan(\gamma_1) - \gamma_2 \tan(\gamma_2)}, \qquad t_2' = \frac{-\gamma_2 \tan(\gamma_2)}{\gamma_1 \tan(\gamma_1) - \gamma_2 \tan(\gamma_2)}. \tag{35}$$

The evolution operator, at the time $\tau = 6T$ where the application of the magnetic field ends, thus becomes:

$$U(\tau = 6T) = \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}T'\right) = \exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}\tau\chi\right), \tag{36}$$

where the effective time $T' = \tau\chi = 6T\chi$ depends on the distortion parameter χ, which in turn depends on the angles γ_1 and γ_2 in the following way:

$$\chi = \frac{1}{3} + \frac{2}{3}\cos^2(\gamma_2)\frac{\tan^2(\gamma_1) - \tan^2(\gamma_2)}{\gamma_1 \tan(\gamma_1) - \gamma_2 \tan(\gamma_2)}. \tag{37}$$

Notice that the required restrictions $t_1' > 0$ and $t_2' > 0$ are satisfied for $n\pi < \gamma_1 < (n+1/2)\pi$ and $(m-1/2)\pi < \gamma_2 < m\pi$, or for $(m-1/2)\pi < \gamma_1 < m\pi$ and $n\pi < \gamma_2 < (n+1/2)\pi$, $m, n \in \mathbf{Z}^+$. Moreover, depending on the values taken by χ in the admissible domain of (γ_1, γ_2), three physically different situations arise:

$$\begin{cases} \chi > 1 & \text{accelerating the free evolution} \\ 0 \leq \chi \leq 1 & \text{slowing the free evolution} \\ \chi < 0 & \text{inverting the free evolution} \end{cases} \tag{38}$$

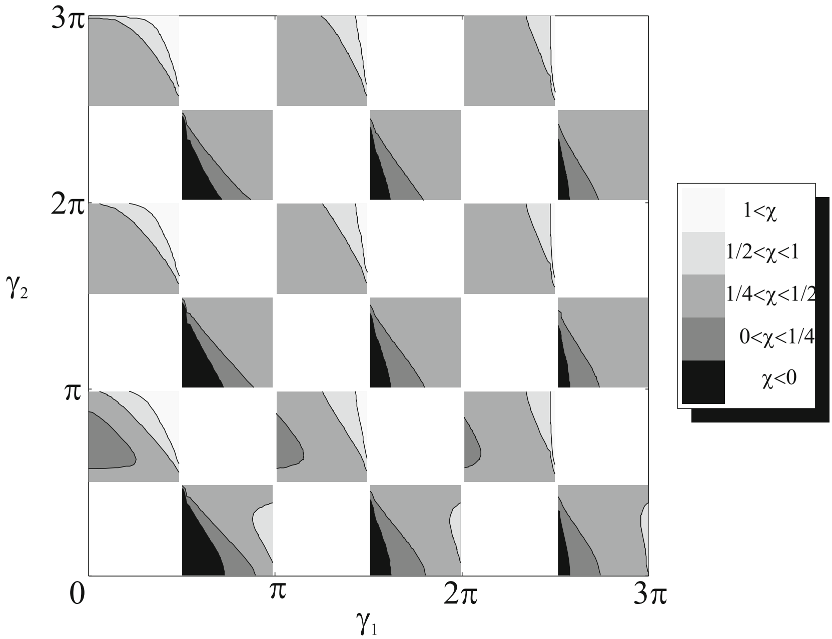

FIGURE 11. The *chessboard of distorted time*, where the manipulated free evolution is induced for a charged particle in the magnetic field of equations (30)–(31).

A plot summarizing these results is shown in Figure 11 [32]. In particular, it is worth noticing the existence of regions where the inversion of the natural free evolution is produced (see the domain in which $\chi < 0$).

A photograph of Bogdan Mielnik, at the time of elaborating [30], is shown in Figure 12.

I have illustrated here, with the example of the free evolution, the way in which we learned to implement the dynamical manipulation. Of course, there are other unitary operations which have been of interest for our group. In particular, it is worth mentioning the squeezing operations, which were explored in detail by Francisco Delgado during the elaboration of his MSc and PhD Thesis [34], under the supervision of Mielnik as well (see also [35–37]). A photograph of both, taken after the MSc dissertation of Francisco Delgado, February 2nd, 1992, is shown in Figure 13.

On the other hand, with Sara Cruz the possible physical meaning of the Floquet operator and its usefulness to achieve several interesting unitary operators was explored in [38]. As a result of its originality and the large amount of new facts contained in Sara's Thesis [39–41] (also under BM supervision), in April 2006 she was awarded the 2005 *Arturo Rosenblueth prize* to the best PhD thesis written at Cinvestav (see photograph in Figure 14).

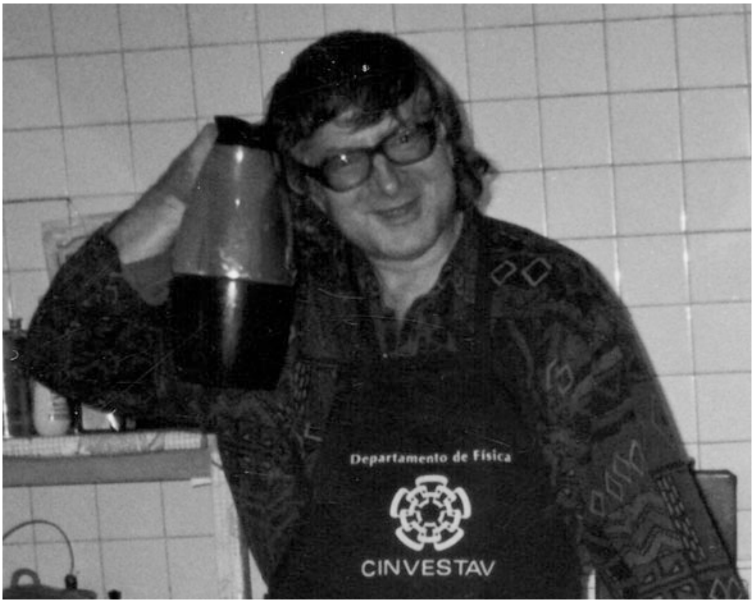

FIGURE 12. The tea time at Paranagua 42-4 in 1993, during some discussions on magnetic control.

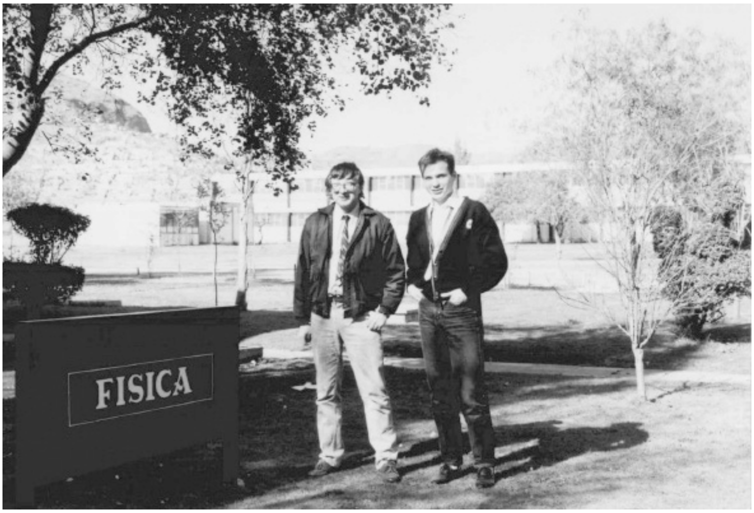

FIGURE 13. Bogdan Mielnik and Francisco Delgado, in front of Physics Department, Cinvestav (February 2nd, 1992).

FIGURE 14. Sara Cruz was awarded (April, 2006) the 2005 *Arturo Rosenblueth Prize* to the best PhD Thesis defended at Cinvestav during the year 2005 (here with her Thesis adviser, Bogdan Mielnik).

In addition, after studying physical problems leading to indefinite Hilbert spaces and non-hermitian Hamiltonians [42], Bogdan Mielnik and Alejandra Ramírez have explored some non-commutative coordinate operators naturally arising when dealing with a charged particle interacting with several magnetic field configurations [43]. Their most recent papers [24, 44, 45] contain several results reported in Alejandra's Thesis. A photograph of Alejandra Ramírez and Bogdan Mielnik, during their participation at the *XXII Workshop on Geometrical Methods in Physics* which was held at Białowieża in 2003 is shown in Figure 15.

It is important to notice that the school of quantum control at Cinvestav also involves colleagues who only did their MSc Thesis under Mielnik's advice. Although they obtained PhDs later on in other areas, we guess that they still conserve some interdisciplinary spirit. I would like to mention specifically:

Gerardo Herrera (September 1987): his MSc Thesis has to do with the dynamical manipulation of a one-dimensional Schrödinger particle in a quadratic potential with a time-dependent frequency [46], the corresponding Hamiltonian is given in equation (20). It was shown that the parity, scale and Fourier transformations can be dynamically induced. A photograph taken after the presentation of the documentary about Plebański's life, January 26th, 2005, containing Gerardo Herrera (at the time Head of the Physics Department), Rosalinda Contreras (then Director of Cinvestav) and Bogdan Mielnik, is shown in Figure 16.

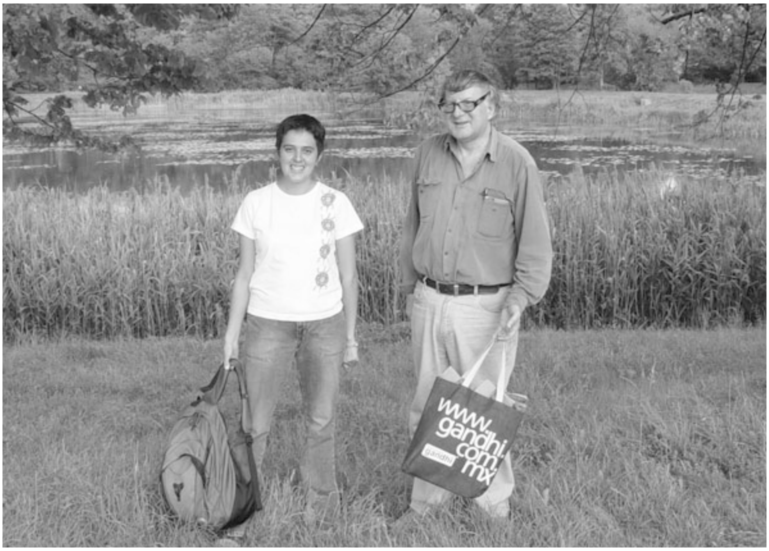

FIGURE 15. Alejandra Ramírez and Bogdan Mielnik at Białowieża, June 2003.

FIGURE 16. Gerardo Herrera (Head of the Physics Department), Rosalinda Contreras (Director of Cinvestav) and Bogdan Mielnik in front of the Arturo Rosenblueth Auditorium, Cinvestav, January 26th, 2005.

Diego Sanjinés (May 23th, 1990): his MSc Thesis addresses the connection between the stationary one-dimensional Schrödinger equation and the classical dynamical problem of an oscillator with time-dependent frequency [47]. It was shown that the stability of the classical problem is closely related to the quantum mechanical problem of eigenvalues.

Francisco Solis (August 1990): in this MSc Thesis, the stability of the motion of charged particles in the vicinity of the nodal points of a monochromatic standing plane wave was analyzed [48]. These results were compared with those obtained from the ponderomotrix potential of Kapitza and Landau.

Marco Antonio Reyes (February 1992): his MSc Thesis has to do with a non-perturbative numerical approach designed for calculating the energy levels of one-dimensional or spherically symmetric potentials [49]. The method was implemented with the angular form of the Riccati equation as a starting point [50].

Since one of the aims of this Conference was to celebrate the 70th and 75th Birthdays of Lech Woronowicz and Bogdan Mielnik respectively, I find interesting to show a photograph (see Figure 17) containing both, along with Jerzy Plebański, at the *Conference to celebrate the Jerzy Plebański's* 75*th Birthday* which was held in Mexico City in September of 2002 [51].

FIGURE 17. Lech Woronowicz, Bogdan Mielnik and Jerzy Plebański (September of 2002).

FIGURE 18. The group of quantum control at the Physics Department of Cinvestav (December 8th, 2010). From left to right and top to bottom: Bogdan Mielnik, Rodrigo Muñoz (upper row); David Bermúdez, Nicolás Fernández, Oscar Rosas, Alonso Contreras, David Fernández, Encarnación Salinas (second row); Sara Cruz, Iván Cabrera, Gerardo Herrera (front row).

In order to provide a global view of Mielnik's scientific work, I would like to close this section by mentioning that he has also contributed substantially to a better understanding of conceptual and polemic problems in quantum mechanics [37, 52–56].

4. Conclusions

By means of these specific examples I have tried to illustrate the way in which we approach the problem of quantum control at the Physics Department of Cinvestav. It is always difficult to evaluate which scientific results will turn important for the future theories or applications, but undoubtedly Professor Bogdan Mielnik has been quite essential in growing up a quantum control group which we hope can compete on the international arena (see Figure 18). On behalf of the group, I would like to express our best wishes to him:

For teaching us the way of doing science.
For teaching us that work has to be done patiently and carefully.
For an atmosphere of permanent creation and discussion.
Long live Professor Bogdan Mielnik!

Acknowledgment

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Partial Inner Product Spaces, a Unifying Language for Quantum Mechanics

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Abstract. In order to obtain a rigorous version of the Dirac formulation of quantum mechanics, one has to go beyond Hilbert space and one usually resorts to a Rigged Hilbert space (RHS). However, this is a particular case of *partial inner product spaces* (PIP-spaces), a general formalism that also generalizes many families of function spaces that play a central role in analysis. In this paper, we shall give an overview of PIP-spaces and operators on them, defined globally. We will also discuss a number of operator classes, such as morphisms and projections.

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1. Introduction

When dealing with singular functions, one generally turns to distributions, most often to tempered distributions. In the latter case, one is in fact working in the triplet (Rigged Hilbert space or RHS)

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mathrm{d}x) \subset \mathcal{S}^\times(\mathbb{R}), \tag{1}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of smooth functions of fast decay and $\mathcal{S}^\times(\mathbb{R})$ is the space of tempered distributions, taken as antilinear continuous functionals over $\mathcal{S}(\mathbb{R})$, so that the embeddings in (1) are linear (we restrict ourselves to one dimension, for simplicity, but the argument is general).

The problem with the triplet (1) is that, besides the Hilbert space vectors, it contains only two types of elements, "very good" functions in $\mathcal{S}$ and "very bad" ones in $\mathcal{S}^\times$. To get a fine control on the behavior of individual elements, one has to interpolate somehow between the two extreme spaces, thus getting a chain of intermediate spaces (see [1] for the Schwartz case).

In fact, this is not at all an isolated case. Indeed many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The typical structure is a *chain of Hilbert or (reflexive) Banach spaces*. Let us give two familiar examples.

(i) The Lebesgue L^p spaces on $[0,1]$, $\mathcal{I} = \{L^p([0,1], \mathrm{d}x),\ 1 \leqslant p \leqslant \infty\}$; here the L^2 inner product $\langle f|g\rangle$ is *not* well defined for two arbitrary functions $f, g \in L^1$. Thus, on L^1, $\langle\cdot|\cdot\rangle$ defines only a *partial* inner product.

(ii) The scale of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ built on the powers of a positive self-adjoint operator $A \geq 1$ in a Hilbert space $\mathcal{H}_0$. Here $\mathcal{H}_n$ is $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual). In this case also, the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathcal{H}_{-\infty}(A) := \bigcup_n \mathcal{H}_n$ it yields only a *partial* inner product.

Thus, in both cases, one gets a *partial inner product space* (PIP-space). In this paper, we shall give a quick overview of PIP-spaces and operators on them. A complete information may be found in the monograph [2].

2. Partial inner product spaces

2.1. Basic definitions

The basic question may be stated as follows: Given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *linear compatibility* [3, 2], by which we mean a symmetric binary relation $\#$ on V which preserves linearity:

$$f\#g \iff g\#f,\ \forall\, f,g \in V,$$
$$f\#g, f\#h \Longrightarrow f\#(\alpha g + \beta h),\ \forall\, f,g,h \in V, \forall\, \alpha,\beta \in \mathbb{C}.$$

As a consequence, for any subset $S \subset V$, the set $S^\# = \{g \in V : g\#f, \forall\, f \in S\}$ is a vector subspace of V and one has $S^{\#\#} = (S^\#)^\# \supseteq S$, $S^{\#\#\#} = S^\#$. Thus one gets the following equivalences:

$$f\#g \iff f \in \{g\}^\# \iff \{f\}^{\#\#} \subseteq \{g\}^\#. \tag{2}$$

From now on, we will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V, \#)$ the family of all assaying subsets of V, ordered by inclusion. Let F be the isomorphy class of $\mathcal{F}$, that is, $\mathcal{F}$ considered as an abstract partially ordered set. Elements of F will be denoted by $r, q, \ldots$, and the corresponding assaying subsets $V_r, V_q, \ldots$. By definition, $q \leq r$ if and only if $V_q \subseteqq V_r$. We also write $V_{\overline{r}} = V_r^\#$, $r \in F$. Thus the relations (2) mean that $f\#g$ if and only if there is an index $r \in F$ such that $f \in V_r$, $g \in V_{\overline{r}}$. In other words, vectors should not be considered individually, but only in terms of assaying subspaces, which are the building blocks of the whole structure.

It is easy to see that the map $S \mapsto S^{\#\#}$ is a closure, in the sense of universal algebra, so that the assaying subspaces are precisely the "closed" subsets. Therefore one has the following standard result [4].

Theorem. *The family* $\mathcal{F}(V,\#) = \{V_r, r \in F\}$ *of all assaying subspaces, ordered by inclusion, is a complete involutive lattice, under the following operations, arbitrarily iterated:*

- *involution:* $V_r \leftrightarrow V_{\overline{r}} := (V_r)^{\#}$,
- *infimum:* $V_{p\wedge q} := V_p \wedge V_q = V_p \cap V_q$, $(p, q, r \in F)$
- *supremum:* $V_{p\vee q} := V_p \vee V_q = (V_p + V_q)^{\#\#}$.

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. By definition, the index set F is also a complete involutive lattice; for instance,

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

A *partial inner product* on $(V, \#)$ is a hermitian form $\langle\cdot|\cdot\rangle$, not necessarily positive definite, defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

The partial inner product clearly defines a notion of *orthogonality*: $f \perp g$ if and only if $f\#g$ and $\langle f|g\rangle = 0$. We require the partial inner product to be *non degenerate*, that is, $(V^{\#})^{\perp} = \{0\}$, i.e., $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. As a consequence, $(V^{\#}, V)$ is a dual pair in the sense of topological vector spaces [5, 6], and so is every couple $(V_r, V_{\overline{r}})$, $r \in F$. In the sequel, we also assume that the partial inner product is positive definite.

Next, one wants the topological structure to match the algebraic structure. Thus the topology $\mathsf{t}(V_r)$ of V_r should be such that the dual of V_r is precisely $V_{\overline{r}}$, that is, $\mathsf{t}(V_r)$ must be a topology of the dual pair $\langle V_r, V_{\overline{r}}\rangle$. Therefore $\mathsf{t}(V_r)$ must be finer than the weak topology $\sigma(V_r, V_{\overline{r}})$ and coarser than the Mackey topology $\tau(V_r, V_{\overline{r}})$:

$$\sigma(V_r, V_{\overline{r}}) \preceq \mathsf{t}(V_r) \preceq \tau(V_r, V_{\overline{r}}).$$

From now on, we assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$. As a consequence, if $V_r[\mathsf{t}(V_r)]$ is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

Typical examples are the following:

(1) Let V be the space ω of *all* complex sequences $x = (x_n)$ and define on it (i) a compatibility relation by $x\#y \Leftrightarrow \sum_{n=1}^{\infty} |x_n\, y_n| < \infty$; (ii) a partial inner product $\langle x|y\rangle = \sum_{n=1}^{\infty} \overline{x_n}\, y_n$, for $x\#y$. Then $\omega^{\#} = \varphi$, the space of finite sequences, and ℓ^2 is the unique central, self-dual, Hilbert space.

(2) Let now V be $L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}x)$, the space of Lebesgue measurable functions, integrable over compact subsets. Define a compatibility relation on it by $f\#g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)|\, \mathrm{d}x < \infty$ and a partial inner product $\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\, \mathrm{d}x$, for $f\#g$. Then $V^{\#} = L^{\infty}_{\mathrm{c}}(\mathbb{R})$, the space of bounded measurable functions of compact support and the central, self-dual, Hilbert space is $L^2(\mathbb{R}, \mathrm{d}x)$.

2.2. Lattices of Hilbert or Banach spaces

The previous examples show that $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, non metrizable spaces, etc. This situation suggests to choose a sublattice $\mathcal{I}$ of $\mathcal{F}$, indexed by I, such that

(i) $\mathcal{I}$ is *generating*, that is, $f\#g$ iff $\exists\, r \in I$ such that $f \in V_r, g \in V_{\overline{r}}$;
(ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;
(iii) $\mathcal{I}$ contains a unique self-dual Hilbert space $V_o = V_{\overline{o}}$.

In that case, the structure $V_I := (V, \mathcal{I}, \langle\cdot|\cdot\rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Both types are particular cases of the so-called indexed PIP-spaces [7], but they are sufficient for our present purposes. Note that $V^{\#}, V$ themselves usually do *not* belong to the family $\{V_r,\, r \in I\}$, but they can be recovered as $V^{\#} = \bigcap_{r\in I} V_r$, $V = \sum_{r\in I} V_r$.

In the LBS case, the lattice structure takes the following form

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm $\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q$;
- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h} \left(\|g\|_p + \|h\|_q\right), g \in V_p, f \in V_q.$$

These norms are usual in interpolation theory [8]. In the LHS case, one takes similar definitions with squared norms, in order to get Hilbert norms.

This construction raises the question of comparison between two compatibilities, thus two PIP-space structures, on the same vector space. Coarsening of a compatibility means selecting an involutive sublattice of $\mathcal{F}$. But the refinement of a compatibility is not always possible, in any case, there is no canonical solution. For a LHS, however, one may proceed by interpolation or with the spectral theorem for self-adjoint operators. An important case is that of refining a RHS into a LHS, e.g., the Schwartz triplet $\mathcal{S} \subset L^2 \subset \mathcal{S}^\times$.

2.3. Examples

We list a series of concrete examples of LBSs (in one dimension, for simplicity). The first two are those described in the Introduction

(i) Scales of Hilbert or Banach spaces

(a) The Lebesgue L^p spaces on $[0,1]$, $\mathcal{I} = \{L^p([0,1], \mathrm{d}x),\ 1 \leqslant p \leqslant \infty\}$.
(b) The scale of Hilbert spaces built on the powers of a positive self-adjoint operator $A \geq 1$ in a Hilbert space $\mathcal{H}_0$. The following examples, all three in $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}x)$ are standard:
   - $(A_{\rm p} f)(x) = (1 + x^2)^{1/2} f(x)$;
   - $(A_{\rm m} f)(x) = (1 - \frac{d^2}{dx^2})^{1/2} f(x)$;
   - $(A_{\rm osc} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$.

In the case of $A_{\rm m}$, the intermediate spaces are the Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$. Note that both $\mathcal{H}_\infty(A_{\rm p}) \cap \mathcal{H}_\infty(A_{\rm m})$ and $\mathcal{H}_\infty(A_{\rm osc})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathcal{H}_{-\infty}(A_{\rm osc})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.

(ii) Sequence spaces

In ω, we may take the lattice $\mathcal{I} = \{\ell^2(r)\}$ of the weighted Hilbert spaces defined as

$$\ell^2(r) = \left\{(x_n) : \sum_{n=1}^{\infty} |x_n|^2\, r_n^{-1} < \infty\right\},\ r = (r_n),\ r_n > 0,$$

with the lattice operations

- infimum: $\ell^2(p \wedge q) = \ell^2(p) \wedge \ell^2(q) = \ell^2(r),\ r_n = \min(p_n, q_n)$,
- supremum: $\ell^2(p \vee q) = \ell^2(p) \vee \ell^2(q) = \ell^2(s),\ s_n = \max(p_n, q_n)$,
- involution: $\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^\times,\ \overline{r}_n = 1/r_n$.

(these norms are equivalent to the projective, resp. inductive norms).

(iii) Spaces of locally integrable functions

In $L^1_{\rm loc}(\mathbb{R}, {\rm d}x)$, we may take the lattice $\mathcal{I} = \{L^2(r)\}$ of the weighted Hilbert spaces defined as

$$L^2(r) = \left\{f \in L^1_{\rm loc}(\mathbb{R}, dx) : \int_{\mathbb{R}} |f(x)|^2\, r(x)^{-1}\, {\rm d}x < \infty\right\},$$

with $r, r^{-1} \in L^2_{\rm loc}(\mathbb{R}, dx), r(x) > 0$ a.e. and the lattice operations

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r),\ r(x) = \min(p(x), q(x))$,
- supremum: $L^2(p \vee q) = L^2(p) \vee L^2(q) = L^2(s),\ s(x) = \max(p(x), q(x))$,
- involution: $L^2(r) \leftrightarrow L^2(\overline{r}),\ \overline{r} = 1/r$.

(iv) The spaces $L^p(\mathbb{R}, {\rm d}x)$, $1 \leqslant p \leqslant \infty$

The spaces $L^p(\mathbb{R}, {\rm d}x)$, $1 \leqslant p \leqslant \infty$, do not constitute a scale, since one has only the inclusions $L^p \cap L^q \subset L^s$, $p < s < q$. Thus one has to consider the lattice they generate, with the following lattice operations:

- $L^p \wedge L^q = L^p \cap L^q$, with the projective norm;
- $L^p \vee L^q = L^p + L^q$, with the inductive norm;
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive Banach spaces and $(L^p \wedge L^q)^\times = L^{\overline{p}} \vee L^{\overline{q}}$, $(L^p \vee L^q)^\times = L^{\overline{p}} \wedge L^{\overline{q}}$.

Thus one gets a genuine lattice of Banach spaces, reflexive for $1 < p, q < \infty$.

3. Operators on PIP-spaces

3.1. Basic idea

As already mentioned, the basic idea of (indexed) PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r ($r \in F$ or $r \in I$), the building blocks of the structure. Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only bounded operators between Hilbert or Banach spaces are allowed. Thus we state:

Given a LHS or LBS $V_I = \{V_r, r \in I\}$, an *operator* on V_I is a map A from a subset $\mathcal{D} \subseteq V$ into V, where

(i) $\mathcal{D}$ is a nonempty union of assaying subsets of V_I;

(ii) for every assaying subset V_q contained in $\mathcal{D}$, there exists a $p \in I$ such that the restriction A_{pq} of A to V_q is linear and continuous into V_p;

(iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathsf{j}(A) := \{(q, p) \in I \times I : A_{pq}$ exists$\}$. Thus the operator A may be identified with the (coherent) collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q, p) \in \mathsf{j}(A)\}.$$

We also need the two sets

$$\mathsf{d}(A) := \{q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\},$$
$$\mathsf{i}(A) := \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}.$$

The following properties are immediate:

- $\mathsf{d}(A)$ is an initial subset of I: if $q \in \mathsf{d}(A)$ and $q' < q$, then $q' \in \mathsf{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator.
- $\mathsf{i}(A)$ is a final subset of I: if $p \in \mathsf{i}(A)$ and $p' > p$, then $p' \in \mathsf{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.
- $\mathsf{j}(A) \subset \mathsf{d}(A) \times \mathsf{i}(A)$, with strict inclusion in general.

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. A similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

3.2. Algebraic operations on operators

Since $V^{\#}$ is dense in V_r, for every $r \in I$, an operator may be identified with a separately continuous sesquilinear form on $V^{\#} \times V^{\#}$. Equivalently, an operator may be identified with a continuous linear map from $V^{\#}$ into V. But the idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint $A^\times$:* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$:

$$\langle A^\times x|y\rangle = \langle x|Ay\rangle, \text{ for } y \in V_r,\ r \in \mathsf{d}(A), \text{ and } x \in V_{\overline{s}},\ s \in \mathsf{i}(A),$$

that is, $(A^\times)_{\overline{rs}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint).

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of the definition.

(ii) *Partial multiplication:* AB is defined if and only if there is a $q \in \mathsf{i}(B) \cap \mathsf{d}(A)$, that is, if and only if there is a continuous factorization through some V_q:

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq}B_{qr}\,.$$

It is worth noting that, for a LHS/LBS, the domain $\mathcal{D}$ of an operator is always a vector subspace of V (this is not true for a general PIP-space).

4. Special classes of operators on PIP-spaces

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHS.

4.1. Homomorphisms

An operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

(i) for every $r \in I$ there exists $u \in K$ such that both A_{ur} and $A_{\overline{ur}}$ exist;
(ii) for every $u \in K$ there exists $r \in I$ such that both A_{ur} and $A_{\overline{ur}}$ exist.

We denote by $\mathrm{Hom}(V_I, Y_K)$ the set of all homomorphisms between the two LHS V_I, Y_K. The following properties are immediate:

(i) $A \in \mathrm{Hom}(V_I, Y_K)$ if and only if $A^\times \in \mathrm{Hom}(Y_K, V_I)$.
(ii) If $A \in \mathrm{Hom}(V_I, Y_K)$, then $\mathsf{j}(A^\times A)$ contains the diagonal of $I \times I$ and $\mathsf{j}(AA^\times)$ contains the diagonal of $K \times K$.
(iii) If $A \in \mathrm{Hom}(V_I)$, then $f\#g$ implies $Af\#Ag$.

A homomorphism $A \in \mathrm{Hom}(V_I, Y_K)$ is an *isomorphism* if there exists $B \in \mathrm{Hom}(Y_K, V_I)$ such that $BA = 1_V, AB = 1_Y$ (identity operators).

An operator U is *unitary* if $U^\times U$ and $UU^\times$ are defined and $U^\times U = 1_V$, $UU^\times = 1_Y$. We emphasize that unitary operators need *not* be homomorphisms! In particular, this implies that the natural setting for group representations in a LHS is that of *unitary isomorphisms* [2, Sec. 3.3.4].

4.2. Symmetric operators

An operator A is *symmetric* if $A^\times = A$. Symmetric operators satisfy a generalized KLMN theorem, stating when a symmetric operator has a self-adjoint restriction to the central Hilbert space V_o [2, Sec. 3.3.5]. Actually, the concept of PIP-space operator allows to treat on the same footing all kinds of operators, from bounded ones to very singular ones. Take, for instance,

$$V_r \subset V_o \simeq V_{\overline{o}} \subset V_s \quad (V_o = \text{Hilbert space}).$$

Then three cases may arise:

- if A_{oo} exists, then A corresponds to a *bounded* operator $V_o \to V_o$;
- if A_{oo} does not exist, but only $A_{or} : V_r \to V_o, r < o$, then A corresponds to an *unbounded* operator, with Hilbert space domain containing V_r;
- if no A_{or} exists, but only $A_{sr} : V_r \to V_s, r < o < s$, then A corresponds to a *singular* operator, with Hilbert space domain possibly reduced to $\{0\}$.

A nice application of this machinery is a rigorous analysis of singular quantum Hamiltonians (e.g., rigorous versions of the Kronig–Penney crystal model or of δ interactions) [2, Sec. 7.1.3].

4.3. Orthogonal projections

An *orthogonal projection* on a non degenerate PIP-space V is a homomorphism that satisfies the relations $P^2 = P = P^\times$ [9]. The set $\mathrm{Proj}(V)$ of all orthogonal projections in V is a partially ordered set, as in a Hilbert space. However, it is a lattice only under additional conditions, yet to be determined. The problem is still open.

These projection operators enjoy several properties similar to those of Hilbert space projectors. Two of them are of special interest here.

(i) Given a non degenerate PIP-space V, there is a natural notion of PIP-subspace, called *orthocomplemented*, which guarantees that such a subspace W of V is again a non degenerate PIP-space with the induced compatibility relation and the restriction of the partial inner product. Then the basic theorem about projections states that a PIP-subspace W of V is orthocomplemented if and only if W is the range of an orthogonal projection $P \in \text{Proj}(V)$, i.e., $W = PV$. Then $V = W \oplus Z$, where Z is another orthocomplemented PIP-subspace.

(ii) An orthogonal projection P is of finite rank if and only if $W = \text{Ran}\, P \subset V^{\#}$ and $W \cap W^{\perp} = \{0\}$. This property has an important consequence for the structure of bases and frames, used for representing and approximating arbitrary elements of a Hilbert space $\mathcal{H}$. Indeed it implies that the basis or frame vectors must belong to the "small" space $V^{\#}$ of any PIP-space V containing $\mathcal{H}$ as central Hilbert space. For the spaces $L^p(\mathbb{R})$, $1 < p < \infty$, e.g., this means that the basis vectors must belong to $V^{\#} = \cap_{1<p<\infty} L^p(\mathbb{R})$. Wavelets, for instance, yield unconditional bases for *all* the spaces $L^p(\mathbb{R})$, $1 < p < \infty$.

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The Resonance-Decay Problem in Quantum Mechanics

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Dedicated to Arno Bohm

Abstract. In the paper the so-called "Resonance-Decay Problem in Quantum Mechanics" is solved for a selected class of Hamiltonians: The absolutely continuous part of the Hamiltonian is unitarily equivalent to a selfadjoint operator H on the Hilbert space $\mathcal{H}_+ := L^2(\mathbb{R}_+, \mathcal{K}, d\lambda), \mathcal{K}$ the multiplicity space, such that H together with the multiplication operator on $\mathcal{H}_+$ forms an asymptotic complete scattering system such that the scattering matrix $S(\cdot)$ is holomorphic in the upper half-plane and satisfies certain conditions at 0, at infinity and on the rim $\mathbb{R}_- + i0$. The proof uses methods of the Lax-Phillips scattering theory.

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1. Introduction

The origin of the resonance-decay problem in non-relativistic quantum mechanics is the observation of bumps in scattering cross-sections. The successful description of a bump by the celebrated Breit-Wigner formula – essentially the modulus of the Breit-Wigner amplitude – with a resonance energy E_0 and a resonance width Γ, where $\Gamma > 0$ is small, suggested first the idea to associate with this *physical resonance* an unstable or decaying state with energy E_0 and lifetime $1/\Gamma$ and with exponential decay law $t \to \mathrm{e}^{-\Gamma t}$. Second, the form of the Breit-Wigner amplitude led to the idea to associate with an hypothetical decaying state of a physical resonance a pole $E_0 - i\Gamma/2$ of the scattering matrix, i.e., a pole in the lower half-plane near the real axis.

The consequence of this idea is that if it turned out that these ideas can be realized then the same properties should be also true for poles with a larger imaginary part, which cannot be identified as a physical resonance, i.e., with a bump.

Therefore, the challenge is to construct the decaying states as eigenstates associated with the poles as corresponding eigenvalues of a non-selfadjoint operator B related to the Hamiltonian. Conversely, the poles should be the only eigenvalues of that operator. That is, this hypothetical operator depends on the scattering operator and it characterizes the set of all poles as its eigenvalue spectrum.

The problem of the construction of an non-selfadjoint operator B, depending on H and characterizing the set of all poles of the scattering matrix as its eigenvalue spectrum, where all eigenvectors satisfy the exponential decay law is called the *resonance-decay problem.*

2. Decay

The deterministic time-evolution of the non relativistic Quantum Mechanics is given by $\mathbb{R} \ni t \to \mathrm{e}^{-itH}\phi$, where ϕ is a normed vector (a state) of a separable Hilbert space $\mathcal{F}$ and where H is the Hamiltonian, a selfadjoint operator on $\mathcal{F}$, which is bounded below. The unitary e^{-itH} is defined by $\mathrm{e}^{-itH} = \int_c^\infty \mathrm{e}^{-it\lambda} E(d\lambda),\ c > -\infty,\ E(\cdot)$ the spectral measure of H. The *transition probability* w.r.t. ϕ and $\mathrm{e}^{-itH}\phi$ is given by $|(\phi, \mathrm{e}^{-itH}\phi)|^2$. A state ϕ is called *decaying* for $t \to \infty$ if $\lim_{t\to\infty}(\phi, \mathrm{e}^{-itH}\phi) = 0$.

The first assumption on H requires that the absolutely continuous spectrum has constant multiplicity and coincides with $[0, \infty)$. Further – for simplicity – it is assumed that H has no singular-continuous spectrum. Then $\mathcal{F} = P^{\mathrm{ac}}\mathcal{F} \oplus \mathcal{E}$, where P^{ac} denotes the projection onto the absolutely continuous subspace of H and $\mathcal{E}$ the closure of all eigenstates. The absolutely continuous part of H, i.e., $H \restriction P^{\mathrm{ac}}\mathcal{F}$, is denoted by H^{ac}.

The lemma of Riemann-Lebesgue implies that all states ϕ from $\mathcal{F}^{\mathrm{ac}}$ are decaying,

$$\lim_{t\to\infty} (\phi, \mathrm{e}^{-itH}\phi) = 0, \quad \phi \in \mathcal{F}^{\mathrm{ac}},$$

whereas the eigenstates of H are stable because $(\phi, \mathrm{e}^{-itH}\phi) = \mathrm{e}^{-it\alpha}$, where $\alpha \geq c$ is a corresponding eigenvalue of ϕ. The function

$$w(t) := |(\phi, \mathrm{e}^{-itH}\phi)|^2, \quad t \geq 0, \quad \phi \in \mathcal{F}^{\mathrm{ac}}, \tag{1}$$

is called its *decay law.*

3. Resonances

In scattering systems in addition to H there appears a second Hamiltonian H_0, in the present context also without singular-continuous spectrum, called the unperturbed one, also defined on $\mathcal{F}$. The second assumption on H requires that the system $\{H, H_0\}$ is *asymptotically complete.* This implies that the absolutely continuous part $H_0^{\mathrm{ac}} := H_0 \restriction P_0^{\mathrm{ac}}\mathcal{F}$ is unitarily equivalent with H^{ac}. Therefore, without restriction of generality, it can be assumed that $P_0^{\mathrm{ac}}\mathcal{F}$ is given by $\mathcal{H}_+ := L^2(\mathbb{R}_+, \mathcal{K}, d\lambda)$, where H_0 acts on this space as the multiplication operator M_+ and $1 \leq \dim \mathcal{K} \leq \infty$ is the multiplicity. That is, the consideration can be based on the spectral representation of H_0^{ac}. Then $\mathcal{F} = \mathcal{H}_+ \oplus \mathcal{E}_0$, where $\mathcal{E}_0$ is the eigenspace of H_0. The unitary equivalence between H^{ac} and H_0^{ac} can be realized by the so-called (isometric) wave operators $W_\pm$ from $\mathcal{H}_+$ onto $P^{\mathrm{ac}}\mathcal{F}$. The (unitary) *scattering operator* $S := W_+^* W_-$ acts on $\mathcal{H}_+$ and the action is realized by the (unitary) operators $\mathbb{R}_+ \ni \lambda \to S(\lambda)$ on $\mathcal{K}$, called *scattering matrix*, via $Sf(\lambda) := S(\lambda)f(\lambda)$.

As already mentioned, bumps in cross-sections of scattering experiments can be approximately described by the *Breit-Wigner formula*, which is essentially given as the modulus of the *Breit-Wigner amplitude*

$$\frac{\Gamma/2}{\lambda - (\lambda_0 - i\Gamma/2)}, \quad \lambda_0 > 0, \quad \Gamma > 0, \quad \Gamma \text{ small.} \tag{2}$$

Eq. (2) suggests to consider the Breit-Wigner amplitude as a term in the scattering amplitude $S(\lambda) - {}_{\mathcal{K}}$, i.e., to conjecture that $\lambda_0 - i\Gamma/2$ could be a *pole* of $S(\cdot)$ in the lower half-plane near the real axis, provided $S(\cdot)$ is analytically continuable. That is, the physical resonances correspond to poles near the real axis, and the conjecture is that for these poles it is possible to construct the mentioned hypothetical decaying states. As it is pointed out before, then the same procedure should be possible also for poles with larger imaginary part, which cannot be identified as a physical resonance. Therefore the following consideration is restricted to scattering systems where the scattering matrix is analytically continuable into the lower half-plane across the positive half-line and for brevity we use the terms *resonance* and *pole of* $S(\cdot)$ synonymously.

In particular as the third assumption the following condition is required:

(i) *The scattering matrix* $S(\cdot)$ *is given by*

$$S(\lambda) = \text{s-lim}_{\epsilon \to +0} S(\lambda + i\epsilon), \quad \lambda > 0, \quad \epsilon > 0,$$

where $\mathbb{C}_+ \ni z \to S(z)$ *is a holomorphic operator function.*

Then $S(\cdot)$ is automatically analytically continuable into $\mathbb{C}_-$ across $\mathbb{R}_+$ by

$$S(z) := (S(\overline{z})^*)^{-1}, \quad z \in \mathbb{C}_-. \tag{3}$$

Thus $S(\cdot)$ is a meromorphic operator function on $\mathbb{C} \setminus (-\infty, 0]$, i.e., in $\mathbb{C}_-$ there are at most *poles* as singularities (examples of scattering systems from potential scattering satisfying condition (i) are considered in [1] and in Reed-Simon [2]). In the following it is assumed that there is at least one pole.

Since it is required that the set of all poles of $S(\cdot)$ coincides with the eigenvalue spectrum of the hypothetical operator B, and further that the corresponding eigenvectors satisfy the exponential decay law, the ansatz is suggested that B is the generator of a so-called decay-semigroup. Then B can be considered as a desired modification of the Hamiltonian and the requirement is satisfied.

Definition. A contractive strongly continuous semigroup $0 \leq t \to \mathrm{e}^{-itA}$ on a Hilbert space $\mathcal{L}$ with $\text{s-lim}_{t\to 0} \mathrm{e}^{-itA} f = f,\ f \in \mathcal{L}$ is called a decay-semigroup if

$$\text{s-lim}_{t\to\infty} \mathrm{e}^{-itA} = 0, \quad f \in \mathcal{L}. \tag{4}$$

This means that, if A has an eigenvalue ζ, then a normed eigenvector $f \in \mathcal{L}$ for this eigenvalue satisfies

$$(f, \mathrm{e}^{-itA} f) = \mathrm{e}^{-it\zeta}. \tag{5}$$

4. Spectral theoretic approach to resonances

In the course of the study of resonances of H – having in mind that one has to identify them as eigenvalues of B – it turned out that there is a close relationship of resonances and eigenvalues of H. In many cases the resonances satisfy the same relations as the eigenvalues – of course via analytic continuation (see, e.g., [1, 3, 4]). Therefore, in this spectral theoretic approach the aim is to characterize resonances as *generalized eigenvalues* of a suitable extension of H^{ac} resp. $\mathrm{e}^{-itH} \restriction \mathcal{F}^{\mathrm{ac}}$ using Gelfand triplets or Rigged Hilbert Spaces (RHS).

The extension approach for $\mathrm{e}^{-itH} \restriction \mathcal{F}^{\mathrm{ac}}$ or, without restriction of generality, for e^{-itM_+} on $\mathcal{H}_+$ is due to Bohm and Gadella (see, e.g., Bohm-Gadella [5], Bohm-Harshman [6], Bohm [7] and further literature, quoted there; for the extension approach for H only see also [8]). Their deductions start with a transfer of the problem from $\mathbb{R}_+$ to $\mathbb{R}$ using the fact that $P_+\mathcal{H}_+^2 \subset \mathcal{H}_+$ is dense in $\mathcal{H}_+$. $P_\pm$ denotes the projections of $\mathcal{H} := L^2(\mathbb{R}, \mathcal{K}, d\lambda)$ given by multiplication with the characteristic function of $\mathbb{R}_\pm$. The subspace $\mathcal{H}_+^2 \subset \mathcal{H}$ is called the *Hardy space* (for the upper half-plane), it is given by $\mathcal{H}_+^2 := FP_-\mathcal{H}$ where F is the Fourier transform. The one has $\mathrm{e}^{-itM} g = \mathrm{e}^{-itM_+} g$ for $g \in P_+\mathcal{H}_+^2$ and $0 \leq t \to \mathrm{e}^{+itM_+}$ is a semigroup on $P_+\mathcal{H}_+^2$. The Gelfand space $\mathcal{G}_+ \subset P_+\mathcal{H}_+^2$ of the RHS used is given by $\mathcal{G}_+ := P_+(\mathcal{H}_+^2 \cap \mathcal{S})$, where $\mathcal{S}$ is the Schwartz space of $\mathcal{H}$. Since P_+ is invertible on $\mathcal{H}_+^2$ it seems – at first sight – that one can work equivalently with $\mathcal{G} := \mathcal{H}_+^2 \cap \mathcal{S} \subset \mathcal{H}_+^2$ itself. $\mathcal{G}$ is invariant w.r.t. the semigroup $0 \leq t \to \mathrm{e}^{itM}$ on $\mathcal{H}$. (Obviously, the whole space $\mathcal{H}_+^2$ is invariant w.r.t. to this semigroup and this is the reason why the introduction of $\mathcal{G}$ in this connection is unnecessary.) The Hardy functions $f \in \mathcal{H}_+^2$ are special continuous antilinear forms on $\mathcal{G}$ and one obtains in this case

$$\begin{aligned} \langle g|(\mathrm{e}^{-itM})^\times f\rangle &= \langle \mathrm{e}^{itM} g, f\rangle = (\mathrm{e}^{itM} g, f) \\ &= (g, \mathrm{e}^{-itM}) = (g, Q_+ \mathrm{e}^{-itM} Q_+ f), \quad g \in \mathcal{G}, \end{aligned}$$

where Q_+ denotes the projection onto $\mathcal{H}_+^2$. This means that for $f \in \mathcal{H}_+^2$ the "extension" of $\mathrm{e}^{-itM} \restriction \mathcal{H}_+^2,\ t \geq 0$ (these operators do not form a semigroup for $t \in \mathbb{R}_+$) is nothing else than the semigroup

$$\mathbb{R}_+ \ni t \to Q_+ \mathrm{e}^{-itM} \restriction \mathcal{H}_+^2 =: C_+(t), \tag{6}$$

which is the adjoint semigroup of the semigroup $0 \leq t \to \mathrm{e}^{itM} \restriction \mathcal{H}_+^2$ (see also [9]).

That is, the evolution e^{-itM} on $\mathcal{H}$ acts for $t \leq 0$ as an isometric semigroup on $\mathcal{H}_+^2$, but for $t \geq 0$ a semigroup action is given by the decay-semigroup $Q_+ \mathrm{e}^{-itM} \restriction \mathcal{H}_+^2$, the adjoint semigroup of the former one. This decay-semigroup is called the *characteristic semigroup* in [10].

The appearance of this semigroup in the course of the deductions of Bohm and Gadella (see [5]–[7]) is an essential step for the solution of the resonance-decay problem:

First, it points to a close connection of this problem to the Lax-Phillips scattering theory (see Lax-Phillips [11]), where the characteristic semigroup plays an important part in the deduction of the famous Lax-Phillips-semigroup. This aspect – the connection with the Lax-Phillips theory – was first emphasized by Y. Strauss (see Strauss [12], see also [10]). Second, the step from $\mathcal{H}_+$ to $\mathcal{H}^2_+$ in the deductions of Bohm and Gadella, which is only motivated by the density of $P_+\mathcal{H}^2_+$ in $\mathcal{H}_+$ raises the question on the status of the problem attained by this step. The spectral theory of the characteristic semigroup is well known

Proposition 1. *The generator C_+ of the characteristic semigroup* (6) *has the following properties:*

(i) res $C_+ = \mathbb{C}_+$,
(ii) *the eigenvalue spectrum of C_+ coincides with $\mathbb{C}_-$,*
(iii) *the eigenspace of the eigenvalue $\zeta \in \mathbb{C}_-$ is given by the subspace*

$$\mathcal{E}_\zeta := \left\{ f \in \mathcal{H}^2_+ : f(z) := \frac{k}{z-\zeta},\ k \in \mathcal{K} \right\} \text{ and one has } C_+(t)f = e^{-it\zeta} f,\ f \in \mathcal{E}_\zeta.$$

For the proof see, e.g., [13].

This means: the spectrum of C_+ contains not only the resonances but the whole lower half-plane, i.e., it contains too many undesired eigenvalues. For example, this point is emphasized in Horwitz-Sigal [14]. A second question refers to the actual meaning of the characteristic semigroup for the resonance-decay problem in the context of $\mathcal{H}_+$. First this requires a transfer of the characteristic semigroup from $\mathcal{H}^2_+$ to $\mathcal{H}_+$.

5. Canonical transfer of the characteristic semigroup to $\mathcal{H}_+$

The semigroup (6) can be canonically transferred from $\mathcal{H}^2_+$ to $\mathcal{H}_+$ by means of the projections P_+ and Q_+ of $\mathcal{H}$.

Since this Hilbert space can be considered as the tensor product of $\mathcal{H}_\mathbb{C}$ and $\mathcal{K}$, i.e., $\mathcal{H} = \mathcal{H}_\mathbb{C} \otimes \mathcal{K}$, where $\mathcal{H}_\mathbb{C} ;= L^2(\mathbb{R}, \mathbb{C}, d\lambda)$ and P_+ and Q_+ act only on the first factor, the operators X considered in the following are always of the form $X = X_\mathbb{C} \otimes \ _\mathcal{K}$.

The polar decomposition of P_+Q_+ reads

$$P_+Q_+ = T^{1/2}R, \tag{7}$$

where $R := \operatorname{sgn}(P_+Q_+)$ is a partial isometry with initial projection Q_+ and final projection P_+ and $T := P_+Q_+P_+$. The operator T is invertible on $\mathcal{H}_+$ and ima $T \subset \mathcal{H}_+$ is dense (see Strauss [14]). These facts are due to the density of $P_+\mathcal{H}^2_+ \subset \mathcal{H}_+$. Note that

$$P_+\mathcal{H}^2_+ = T^{1/2}\mathcal{H}_+,$$

because of

$$P_+ f = P_+ Q_+ f = T^{1/2} R f = T^{1/2} f_+, \quad f_+ = Rf, \quad f \in \mathcal{H}_+^2.$$

Then

$$C_+^R(t) := R\mathrm{e}^{-itM} R^*, \quad t \geq 0 \tag{8}$$

is the transferred semigroup corresponding to $C_+(\cdot)$. Its relation to the evolution $t \to \mathrm{e}^{-itM_+}$ on $\mathcal{H}_+$ is given by

Proposition 2. *The semigroup C_+^R acts on $T^{1/2}\mathcal{H}_+$ by*

$$C_+^R(t) T^{1/2} f = T^{1/2}(e^{-itM_+} f), \quad f \in \mathcal{H}_+. \tag{9}$$

The eigenspace of C_+^R for $\zeta \in \mathbb{C}_-$ is given by $\mathcal{E}_\zeta^+ := R\mathcal{E}_\zeta$.

Proof. One obtains by calculation

$$\begin{aligned} R\mathrm{e}^{-itM} R^* f &= T^{-1/2} P_+ Q_+ \mathrm{e}^{-itM} Q_+ P_+ T^{-1/2} T^{1/2} f \\ &= T^{-1/2} P_+ Q_+ \mathrm{e}^{-itM} Q_+ P_+ f = T^{-1/2} P_+ Q_+ \mathrm{e}^{-itM} P_+ f \\ &= T^{-1/2} P_+ Q_+ P_+ \mathrm{e}^{-itM_+} f = T^{1/2}(\mathrm{e}^{-itM_+} f), \quad f \in \mathcal{H}_+, \end{aligned}$$

because of $Q_+ \mathrm{e}^{-itM} Q_+ = Q_+ \mathrm{e}^{-itM}$. The second assertion is obvious. □

Equation (9) means that the action of e^{-itM_+} can be described by

$$\mathrm{e}^{-itM_+} f = T^{-1/2} C_+^R(t) T^{1/2} f \quad f \in \mathcal{H}_+ \tag{10}$$

Corollary 1. *For all $\zeta \in \mathbb{C}_-$ the intersection of $\mathcal{E}_\zeta^+$ and $T^{1/2}\mathcal{H}_+$ contains only* 0*,* i.e.*,*

$$R\mathcal{E}_\zeta \cap T^{1/2}\mathcal{H}_+ = \{0\}.$$

Proof. Assume that there is an $f \neq 0$ with $f = Re_\zeta$ and $f = T^{1/2} f_+$, $f_+ \in \mathcal{H}_+$. Then $C_+^R(t) f = \mathrm{e}^{-it\zeta} f = \mathrm{e}^{-it\zeta}(T^{1/2} f_+)$. Together with (9) this gives

$$T^{1/2}(\mathrm{e}^{-itM_+} f_+) = T^{1/2}(\mathrm{e}^{-it\zeta} f_+),$$

thus one obtains $\mathrm{e}^{-itM_+} f_+ = \mathrm{e}^{-it\zeta} f_+$, but this contradicts to the unitarity of e^{-itM_+}. □

The decay law of vectors $f_+ \in T^{1/2}\mathcal{H}_+$ is given by

$$t \to |(T^{1/2} f, \mathrm{e}^{-itM_+} T^{1/2} f)|^2, \quad f \in \mathcal{H}_+.$$

According to (9) one has

$$(T^{1/2} f, \mathrm{e}^{-itM_+} T^{1/2} f) = (f, C_+^R(t) T f).$$

This suggests that there is almost no chance to construct vectors from $T^{1/2}\mathcal{H}_+$ such that the corresponding decay law is exactly an exponential one. However, since $T\mathcal{H}_+$ is also dense in $\mathcal{H}_+$ one obtains an approximate exponential decay law if one chooses vectors $T^{1/2} f_+$ such that $T f_+$ is "near" to an eigenvector Re_ζ. where $e_\zeta \in \mathcal{E}_\zeta$.

Corollary 2. *Let $f_+ \in \mathcal{H}_+$ and Tf_+ an approximation for an eigenvector of the transformed characteristic semigroup, i.e., there is an $e_\zeta \in \mathcal{E}_\zeta$* such that $\|Tf_+ - Re_\zeta\| < \epsilon$ *for some $\epsilon > 0$. Then*

$$|(T^{1/2}f_+, e^{-itM_+}T^{1/2}f_+)| \leq \|f_+\|(e^{-t|\operatorname{Im}\zeta|} + \epsilon).$$

Proof. Obvious because of

$$(T^{1/2}f_+, \mathrm{e}^{-itM_+}T^{1/2}f_+) = (f_+, R\mathrm{e}^{-itM}R^*(Tf_+ - Re_\zeta)) + \mathrm{e}^{-it\zeta}(f_+, Re_\zeta). \quad \square$$

Remark. Since Tf_+ is an approximation of Re_ζ, e.g., for $\|e_\zeta\| = 1$, then $\|Tf_+\|^2 = \int_0^1 \lambda^2(f_+, E(d\lambda)f_+)$ is a number a near to 1, where $E(\cdot)$ denotes the spectral measure of T, and $\|f_+\|^2 = \int_0^1 (f_+, E(d\lambda)f_+)$, which can be much larger. That is, the better the approximation, i.e., the smaller ϵ, the larger can be $\|f_+\|$.

6. Time-dependent characterization of the set of all resonances

The first question at the end of Section 4, the characterization of the poles by the characteristic semigroup can be solved by the construction of an suitable invariant subspace depending on the scattering operator.

Every invariant subspace $\mathcal{T} \subset \mathcal{H}_+^2$ for the semigroup $C_+(\cdot)$ defines a sub-semigroup $D_+(\cdot)$ on $\mathcal{T}$:

$$D_+(t) := \mathrm{e}^{-itC_+} \restriction \mathcal{T} = \mathrm{e}^{-itD_+}$$

where D_+ denotes the generator of the restricted semigroup. Obviously spec $D_+ \subseteq$ spec C_+, or res $C_+ \subseteq$ res D_+, i.e., $\mathbb{C}_+ \subseteq$ res D_+, $D_+(\cdot)$ is again contractive and decaying. In particular, the eigenvalue spectrum of D_+ is a subset of $\mathbb{C}_-$.

The decisive idea for the construction of an invariant subspace such that the eigenvalue spectrum coincides with the set of all resonances is to consider the linear manifold $\mathcal{N}_+ \subset \mathcal{H}_+^2$ of all $g \in \mathcal{H}_+^2$ such that

$$\sup_{y>0} \int_{-\infty}^{\infty} \|S(x+iy)g(x+iy)\|_{\mathcal{K}}^2 dx < \infty \tag{11}$$

Then, according to the theorem of Paley-Wiener, the vector-function

$$z \to f(z) := S(z)g(z)$$

defines also an element $f \in \mathcal{H}_+^2$. The set of all such vector-functions is again a linear manifold $\mathcal{M}_+ \subset \mathcal{H}_+^2$ and

$$\mathcal{T}_+ := \mathcal{H}_+^2 \ominus \mathcal{M}_+ \tag{12}$$

is a subspace. Obviously $\mathcal{T}_+ \supset \{0\}$ and one has

Proposition 3. *The subspace $\mathcal{T}_+$ is invariant w.r.t. $C_+(\cdot)$.*

Proof. Let $g \in \mathcal{M}_+$ and $f \in \mathcal{T}_+$. Then

$$(C_+(t)f, g) = (Q_+ \mathrm{e}^{-itM} f, g) = (f, \mathrm{e}^{itM} g)$$

and $\mathrm{e}^{itM} g \in \mathcal{M}_+$ because of $S(z)\mathrm{e}^{itz} g(z) = \mathrm{e}^{itz} S(z) g(z)$. Hence $(f, \mathrm{e}^{itM} g) = 0$ and $C_+(t)f \in \mathcal{T}_+$ for all $t \geq 0$. □

The generator of $C_+(\cdot) \restriction \mathcal{T}_+$ is denoted by B_+. It depends on the scattering operator S.

In order to obtain a smooth result, in the following it is assumed that

$$\dim \mathcal{K} < \infty. \tag{13}$$

The reason is that in this case $\zeta \in \mathbb{C}_-$ is a pole of $S(\cdot)$ if and only if $\ker\, S(\overline{\zeta})^* \supset \{0\}$.

Proposition 4. *Let* (13) *be true. If ζ is a resonance then it is an eigenvalue of B_+ and the corresponding eigenvectors are the functions f:*

$$f(\lambda) := \frac{k}{\lambda - \zeta}, \quad S(\overline{\zeta})^* k = 0.$$

Proof. First f is an eigenvector of C_+ with eigenvalue ζ. That is, only $f \in \mathcal{T}_+$, i.e., $(f, g) = 0$ for all $g \in \mathcal{M}_+$ is to be proved. One has

$$\begin{aligned}(f, g) &= \int_{-\infty}^{\infty} \left(\frac{k}{\lambda - \zeta}, g(\lambda + i0) \right)_{\mathcal{K}} d\lambda = \int_{-\infty}^{\infty} \frac{1}{\lambda - \overline{\zeta}} (k, g(\lambda + i0))_{\mathcal{K}} d\lambda \\ &= 2\pi i (k, g(\overline{\zeta}))_{\mathcal{K}} = 2\pi i (k, S(\overline{\zeta}) f(\overline{\zeta}))_{\mathcal{K}} d\lambda = 2\pi i (S(\overline{\zeta})^* k, f(\overline{\zeta}))_{\mathcal{K}} = 0.\end{aligned}$$ □

Therefore, the crucial question with regard to the characterization problem of the resonances is under which conditions for S the following statement is true:

Conjecture. *Let $\mathcal{T}_+, B_+$ be as before and assume* (13)*. Then*

(I) *ζ is a resonance iff ζ is an eigenvalue of B_+.*
(II) *$S(\cdot)$ is holomorphic at ζ iff $\zeta \in \operatorname{res} B_+$.*

The following conditions for S are sufficient for the validity of the conjecture.

Theorem. *Let $\mathcal{T}_+, B_+$ be as before. Assume* (13) *and that $S(\cdot)$ satisfies the following additional conditions:*

(i) *$S(\cdot)$ is meromorphic on the rim $\mathbb{R}_- + i0$ and there are at most finitely many poles at $\mathbb{R}_- + i0$,*
(ii) *$S(\cdot)$ is bounded at $z = 0$ and $z = \infty$, i.e.,*

$$\sup_{z \in \mathcal{G}} \|S(z)\| < \infty, \quad \mathcal{G} := \{z \in \mathbb{C}_+ : |z| < r_0, |z| > r\}, \quad 0 < r_0 < r.$$

Then the assertions (I) *and* (II) *of the conjecture are true and moreover one has* $\mathbb{R} \subset \operatorname{res} B_+$.

The proof uses methods of the Lax-Phillips-theory (see [11]). It is given in [10].

Examples for Hamiltonians H such that the conditions (i), (ii) are satisfied are selected trace class perturbations $H := M_+ + V$ (see [10]), also Hamiltonians of potential scattering (see [2]).

Remark. Condition (i) of the theorem implies that a function $g \in \mathcal{M}_+$, given by the function $z \to S(z)f(z), f \in \mathcal{H}^2_+$, in the upper half-plane and defined on $\mathbb{R}$ by $g(\lambda) := \text{s-lim}_{\epsilon \to +0} S(\lambda + i\epsilon)f(\lambda + i\epsilon)$ is nothing else than the function $g(\lambda) = S(\lambda + i0)f(\lambda + i0)$, which is defined almost everywhere on $\mathbb{R}$ where the possible poles are points from the exceptional set. The theorem says that the subspace $\mathcal{T}_+$, defined by (12), equals the closure of the linear span of all eigenvectors $e_{\zeta,k}$ of the characteristic semigroup for the points ζ, which are poles of $S(\cdot)$ and where k satisfies the equation $S(\overline{\zeta})^* k = 0$. If, for example, $\mathcal{N}_+ \subset \mathcal{H}^2_+$ is dense then the condition (12) means simply that $f \in \mathcal{T}_+$ if and only if

$$S^* f \in \mathcal{H}^2_-, \tag{14}$$

where S^* on $\mathbb{R}_- + i0$ is defined by $S^*(\lambda + i0) = S(\lambda + i0)^* = S(\lambda - i0)^{-1}$, i.e., the condition (14) for some $f \in \mathcal{H}^2_+$ is sufficient for the property that f is in the closure of all linear combinations of certain eigenvectors (note the condition for the vectors k) of the characteristic semigroup for the poles of $S(\cdot)$.

For the scattering operators S with the properties (i), (ii) this theorem presents a solution of the resonance-decay problem: The "non-selfadjoint operator B related to H" required in the formulation of the problem is the generator of the constructed restriction of the transformed characteristic semigroup to the subspace $R\mathcal{T}_+$, where $\mathcal{T}_+$ is defined by the conditions (11) and (12).

Concerning the calculation of the normed eigenvectors for ζ, given by

$$f_\zeta(\lambda) := R\left(\frac{k}{\cdot - \zeta}\right)(\lambda),$$

where $\|f_\zeta\| = \|k\|_{\mathcal{K}} \left(\frac{|\mathrm{Im}\ \zeta|}{\pi}\right)^{1/2}$, note that

$$Tf(\lambda) = (P_+ Q_+ P_+)f(\lambda) = \frac{1}{2\pi i}\chi_{\mathbb{R}_+}(\lambda)\int_0^\infty \frac{f(\mu)}{\mu - (\lambda + i0)}d\mu.$$

This is a positive operator and since $Rg = T^{-1/2}P_+ g,\ g \in \mathcal{H}^2_+$, in order to get f_ζ one has to solve the operator equation

$$(T^{1/2}f_\zeta)(\lambda) = \chi_{\mathbb{R}_+}(\lambda)\frac{k}{\lambda - \zeta}.$$

This points to the problem to calculate the spectral measure of T resp. the corresponding "generalized eigenfunctions" of this operator.

7. Conclusion

First of interest is the question for further sufficient conditions such that the conjecture is true, also the investigation of the case that the multiplicity space is infinite-dimensional.

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Geometry of the Set of Mixed Quantum States: An Apophatic Approach

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Dedicated to prof. Bogdan Mielnik on the occasion of his 75th birthday

Abstract. The set of quantum states consists of density matrices of order N, which are hermitian, positive and normalized by the trace condition. We analyze the structure of this set in the framework of the Euclidean geometry naturally arising in the space of hermitian matrices. For $N = 2$ this set is the Bloch ball, embedded in ^{3}. For $N \geq 3$ this set of dimensionality $N^2 - 1$ has a much richer structure. We study its properties and at first advocate an apophatic approach, which concentrates on characteristics not possessed by this set. We also apply more constructive techniques and analyze two-dimensional cross-sections and projections of the set of quantum states. They are dual to each other. At the end we make some remarks on certain dimension dependent properties.

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1. Introduction

Quantum information processing differs significantly from processing of classical information. This is due to the fact that the space of all states allowed in the quantum theory is much richer than the space of classical states [1, 2, 3, 4, 5, 6]. Thus an author of a quantum algorithm, writing a screenplay designed specially for the quantum scene, can rely on states and transformations not admitted by the classical theory.

For instance, in the theory of classical information the standard operation of inversion of a bit, called the NOT gate, cannot be represented as a concatenation of two identical operations on a bit. But the quantum theory allows one to construct the gate called $\sqrt{\text{NOT}}$, which performed twice is equivalent to the flip of a qubit.

This simple example can be explained by comparing the geometries of classical and quantum state spaces. Consider a system containing N perfectly distinguishable states. In the classical case the set of classical states, equivalent to N-point probability distributions, forms a regular simplex Δ_{N-1} in $N-1$ dimensions. Hence the set of pure classical states consists of N isolated points. In a quantum set-up the set of states $\mathcal{Q}_N$, consisting of hermitian, positive and normalized density matrices, has N^2-1 real dimensions. Furthermore, the set of pure quantum states is connected, and for any two pure states there exist transformations that take us along a continuous path joining the two quantum pure states. This fact is one of the key differences between the classical and the quantum theories [7].

The main goal of the present work is to provide an easy-to-read description of similarities and differences between the sets of classical and quantum states. Already when $N=3$ the geometric structure of the eight-dimensional set $\mathcal{Q}_3$ is not easy to analyse nor to describe [8, 9]. Therefore we are going to use an *apophatic approach*, in which one tries to describe the properties of a given object by specifying simple features it *does not* have. Then we use a more conventional [10, 11, 12] constructive approach and investigate two-dimensional cross-sections and projections of the set $\mathcal{Q}_3$ [13, 14, 15]. Thereby a cross-section is defined as the intersection of a given set with an affine space. We happily recommend a very recent work for a more exhaustive discussion of the cross-sections [16].

2. Classical and quantum states

A classical state is a probability vector $\vec{p}=(p_1,p_2,\ldots,p_N)^{\mathrm{T}}$, such that $\sum_i p_i=1$ and $p_i\geq 0$ for $i=1,\ldots,N$. Assuming that a pure quantum state $|\psi\rangle$ belongs to an N-dimensional Hilbert space $\mathcal{H}_N$, a general quantum state is a density matrix ρ of size N, which is hermitian, $\rho=\rho^\dagger$, with positive eigenvalues, $\rho\geq 0$, and normalized, $\mathrm{Tr}\rho=1$. Note that any density matrix can be diagonalised, and then it has a probability vector along its diagonal. But clearly the space of all quantum states $\mathcal{Q}_N$ is significantly larger than the space of all classical states – there are $N-1$ free parameters in the probability vector, but there are N^2-1 free parameters in the density matrix.

The space of states, classical or quantum, is always a *convex* set. By definition a convex set is a subset of Euclidean space, such that given any two points in the subset the line segment between the two points also belongs to that subset. The points in the interior of the line segment are said to be *mixtures* of the original points. Points that cannot be written as mixtures of two distinct points are called *extremal* or *pure*. Taking all mixtures of three pure points we get a triangle Δ_2, mixtures of four pure points form a tetrahedron Δ_3, etc.

The individuality of a convex set is expressed on its boundary. Each point on the boundary belongs to a *face*, which is in itself a convex subset. To qualify as a face this convex subset must also be such that for all possible ways of decomposing any of its points into pure states, these pure states themselves belong to the subset.

We will see that the boundary of $\mathcal{Q}_N$ is quite different from the boundary of the set of classical states.

2.1. Classical case: the probability simplex

The simplest convex body one can think of is a *simplex* Δ_{N-1} with N pure states at its corners. The set of all classical states forms such a simplex, with the probabilities p_i telling us how much of the ith pure state that has been mixed in. The simplex is the only convex set which is such that a given point can be written as a mixture of pure states in one and only one way.

The number r of non-zero components of the vector $\vec{p}$ is called the rank of the state. A state of rank one is pure and corresponds to a corner of the simplex. Any point inside the simplex Δ_{N-1} has full rank, $r = N$. The boundary of the set of classical states is formed by states with rank smaller than N. Each face is itself a simplex Δ_{r-1}. Corners and edges are special cases of faces. A face of dimension one less than that of the set itself is called a *facet.*

It is natural to think of the simplex as a regular simplex, with all its edges having length one. This can always be achieved, by defining the distance between two probability vectors $\vec{p}$ and $\vec{q}$ as

$$D[\vec{p}, \vec{q}] = \sqrt{\frac{1}{2}\sum_{i=1}^{N}(p_i - q_i)^2}\ . \tag{1}$$

The geometry is that of Euclid. With this geometry in place we can ask for the *outsphere*, the smallest sphere that surrounds the simplex, and the *insphere*, the largest sphere inscribed in it. Let the radius of the outsphere be R_N and that of the insphere be r_N. One finds that $R_N/r_N = N - 1$.

2.2. The Bloch ball

Another simple example of a convex set is a three-dimensional ball. The pure states sit on its surface, and each such point is a zero-dimensional face. There are no higher-dimensional faces (unless we count the entire ball as a face). Given a point that is not pure it is now possible to decompose it in infinitely many ways as a mixture of pure states.

Remarkably this ball is the space of states $\mathcal{Q}_2$ of a single *qubit*, the simplest quantum mechanical state space. For concreteness introduce the Pauli matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. These three matrices form an orthonormal basis for the set of traceless Hermitian matrices of size two, or in other words for the Lie algebra of $SU(2)$. If we add the identity matrix $\sigma_0 = \quad = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we can expand an arbitrary state ρ in this basis as

$$\rho = \frac{1}{2}\quad + \sum_{i=1}^{3}\tau_i\sigma_i\ , \tag{2}$$

where the expansion coefficients are $\tau_i = \mathrm{Tr}\,\rho\sigma_i/2$. These three numbers are real since the matrix ρ is Hermitian. The three-dimensional vector $\vec{\tau} = (\tau_1, \tau_2, \tau_3)^{\mathrm{T}}$

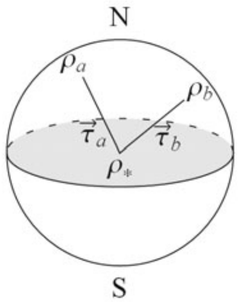


FIGURE 1. The set of mixed states of a qubit forms the *Bloch ball* with pure states at the boundary and the maximally mixed state $\rho_* = \frac{1}{2}$ at its center: The Hilbert–Schmidt distance between any two states is the length of the difference between their Bloch vectors, $||\vec{\tau}_a - \vec{\tau}_b||$.

is called the *Bloch vector* (or coherence vector). If $\vec{\tau} = 0$ we have the *maximally mixed state*. Pure states are represented by projectors, $\rho = \rho^2$.

Since the Pauli matrices are traceless the coefficient $\frac{1}{2}$ standing in front of the identity matrix assures that $\text{Tr}\rho = 1$, but we must also ensure that all eigenvalues are non-negative. By computing the determinant we find that this is so if and only if the length of the Bloch vector is bounded, $||\vec{\tau}||^2 \leq \frac{1}{2}$. Hence $\mathcal{Q}_2$ is indeed a solid ball, with the pure states forming its surface – the *Bloch sphere*.

A simple but important point is that the set of classical states Δ_1, which is just a line segment in this case, sits inside the Bloch ball as one of its diameters. This goes for any diameter, since we are free to regard any two commuting projectors as our classical bit. Two commuting projectors sit at antipodal points on the Bloch sphere. To ensure that the distance between any pair of antipodal equals one we define the distance between two density matrices ρ_A and ρ_B to be

$$D_{\text{HS}}(\rho_A, \rho_B) \;=\; \sqrt{\frac{1}{2}\text{Tr}[(\rho_A - \rho_B)^2]}\;. \tag{3}$$

This is known as the *Hilbert-Schmidt distance*. Let us express this in the Cartesian coordinate system provided by the Bloch vector,

$$D_{\text{HS}}[\rho_A, \rho_B] \;=\; \sqrt{\sum_{i=1}^{3}(\tau_i^A - \tau_i^B)^2} \;=\; ||\vec{\tau}^A - \vec{\tau}^B||\;. \tag{4}$$

This is the Euclidean notion of distance, see Figure 1.

2.3. Quantum case: $\mathcal{Q}_N$

When $N > 2$ the quantum state space is no longer a solid ball. It is always a convex set however. Given two density matrices, that is to say two positive hermitian matrices $\rho, \sigma \in \mathcal{Q}_N$. It is then easy to see that any convex combination of these two states, $a\rho + (1-a)\sigma \in \mathcal{Q}_N$ where $a \in [0,1]$, must be a positive matrix too, and hence it belongs to $\mathcal{Q}_N$. This shows that the set of quantum states is convex. For all N the face structure of the boundary can be discussed in a unified way. Moreover it remains true that $\mathcal{Q}_N$ is swept out by rotating a classical probability simplex Δ_{N-1} in ${}^{N^2-1}$, but for $N > 2$ there are restrictions on the allowed rotations.

To make these properties explicit we start with the observation that any density matrix can be represented as a convex combination of pure states

$$\rho = \sum_{i=1}^{k} p_i \, |\phi_i\rangle\langle\phi_i|, \tag{5}$$

where $\vec{p} = (p_1, p_2, \ldots, p_k)$ is a probability vector. In contrast to the classical case there exist infinitely many decompositions of any mixed state $\rho \neq \rho^2$. The number k can be arbitrarily large, and many different choices can be made for the pure states $|\phi_i\rangle$. But there does exist a distinguished decomposition. Diagonalising the density matrix we find its eigenvalues $\lambda_i \geq 0$ and eigenvectors $|\psi_i\rangle$. This allows us to write the eigendecomposition of a state,

$$\rho = \sum_{j=1}^{N} \lambda_j |\psi_j\rangle\langle\psi_j| \ . \tag{6}$$

The number r of non-zero components of the probability vector $\vec{\lambda}$ is called the rank of the state ρ, and does not exceed N. This is the usual definition of the rank of a matrix, and by happy accident it agrees with the definition of rank in convex set theory: the *rank* of a point in a convex set is the smallest number of pure points needed to form the given point as a mixture.

Consider now a general convex set in d dimensions. Any point belonging to it can be represented by a convex combination of not more than $d+1$ extremal states. Interestingly, $\mathcal{Q}_N$ has a peculiar geometric structure since any given density operator ρ can be represented by a combination of not more than N pure states, which is much smaller than $d + 1 = N^2$. In Hilbert space these N pure states are the orthogonal eigenvectors of ρ. If we adopt the Hilbert-Schmidt definition of distance (3) they form a copy of the classical state space, the regular simplex Δ_{N-1}.

Conversely, every density matrix can be reached from a diagonal density matrix by means of an $SU(N)$ transformation. Such transformations form a subgroup of the rotation group $SO(N^2-1)$. Therefore any density matrix can be obtained by rotating a classical probability simplex around the maximally mixed state, which is left invariant by rotations. However, when $N > 2$ $SU(N)$ is a proper subgroup of $SO(N^2 - 1)$, which is why $\mathcal{Q}_N$ forms a solid ball only if $N = 2$. The relative sizes of the outsphere and the insphere are still related by $R_N/r_N = N - 1$.

The boundary of the set $\mathcal{Q}_N$ shows some similarities with that of its classical cousin. It consists of all matrices whose rank is smaller than N. There will be faces of rank 1 (the pure states), of rank 2 (in themselves they are copies of $\mathcal{Q}_2$), and so on up to faces of rank $N-1$ (copies of $\mathcal{Q}_{N-1}$). Note that there are no hard edges: the minimal non-extremal faces are solid three-dimensional balls. The largest faces

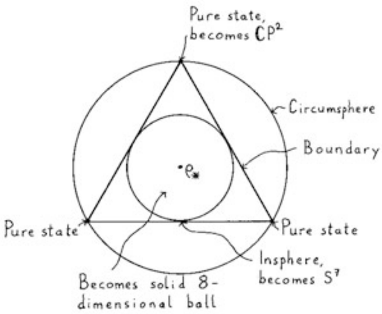


FIGURE 2. The set $\mathcal{Q}_3$ of quantum states of a qutrit contains positive semi-definite matrices with spectrum from the simplex Δ_2 of classical states. The corners of the triangle become the $4D$ set of pure states, the edges lead to the $7D$ boundary $\partial\mathcal{Q}_3$, while interior of the triangle gives the interior of the $8D$ convex body. The set $\mathcal{Q}_3$ is inscribed inside a 7-sphere of radius $R_3 = \sqrt{2/3}$ and it contains an 8-ball of radius $r_3 = 1/\sqrt{6}$.

have a dimension much smaller than the dimension of the boundary of $\mathcal{Q}_N$. As in the classical case, any face can be described as the intersection of the convex set with a bounding hyperplane in the container space. In technical language one says that all faces are exposed. Note also that every point on the boundary belongs to a face that is tangent to the insphere. This has the interesting consequence that the area A of the boundary is related to the volume V of the body by

$$rA/V = d \; , \tag{7}$$

where r is the radius of the insphere and d is the dimension of the body (in this case $d = N^2 - 1$) [17]. This implies that $\mathcal{Q}_N$ has a constant height [17] and can be decomposed into pyramids of equal height having all their apices at the centre of the inscribed sphere. Incidentally the volume of $\mathcal{Q}_N$ is known explicitly [18].

There are differences too. A typical state on the boundary has rank $N - 1$, and any two such states can be connected with a curve of states such that all states on the curve have the same rank. In this sense $\mathcal{Q}_N$ is more like an egg than a polytope [19]. See Figure 2 about spheres and the boundary of $\mathcal{Q}_N$ compared to those of Δ_{N-1}.

We can regard the set of N by N matrices as a vector space (called Hilbert-Schmidt space), endowed with the scalar product

$$\langle A|B\rangle_{\mathrm{HS}} = \tfrac{1}{2}\,\mathrm{Tr}\, A^{\dagger}B \; . \tag{8}$$

The set of hermitian matrices with unit trace is not a vector space as it stands, but it can be made into one by separating out the traceless part. Thus we can represent a density matrix as

$$\rho = \frac{1}{N} \quad + u \; , \tag{9}$$

where u is traceless. The set of traceless matrices is an Euclidean subspace of Hilbert-Schmidt space, and the Hilbert-Schmidt distance (3) arises from this scalar product. In close analogy to equation (2) we can introduce a basis for the set of traceless matrices, and write the density matrix in the *generalized Bloch vector* representation,

$$\rho = \frac{1}{N} \; + \sum_{i=1}^{N^2-1} u_i \gamma_i \; . \tag{10}$$

Here γ_i are hermitian basis vectors. The components u_i must be chosen such that ρ is a positive definite matrix.

2.4. Dual and self-dual convex sets

Both the classical and the quantum state spaces have the remarkable property that they are *self-dual.* But the word duality has many meanings. In projective geometry the dual of a point is a plane. If the point is represented by a vector $\vec{x}$, we can define the dual plane as the set of vectors $\vec{y}$ such that

$$\vec{x} \cdot \vec{y} = -1 \; . \tag{11}$$

The dual of a line is the intersection of a one-parameter family of planes dual to the points on the line. This is in itself a line. The dual of a plane is a point, while the dual of a curved surface is another curved surface – the envelope of the planes that are dual to the points on the original surface. To define the dual of a convex body with a given boundary we change the definition slightly, and include all points on one side of the dual planes in the dual. Thus the *dual* X^* of a convex body X is defined to be

$$X^* = \{\vec{x} \mid 1 + \vec{x} \cdot \vec{y} \; \geq \; 0 \; \forall \vec{y} \in X\} \; . \tag{12}$$

The dual of a convex body including the origin is the intersection of half-spaces $\{\vec{x} \mid 1 + \vec{x} \cdot \vec{y} \geq 0\}$ for extremal points $\vec{y}$ of X [20]. If we enlarge a convex body the conditions on the dual become more stringent, and hence the dual shrinks. The dual of a sphere centred at the origin is again a sphere, so a sphere (of suitable radius) is self-dual. The dual of a cube is an octahedron. The dual of a regular tetrahedron is another copy of the original tetrahedron, possibly of a different size. Hence this is a self-dual body. Convex subsets $F \subset X$ are mapped to subsets of X^* by

$$F \; \mapsto \; \widehat{F} := \{\vec{x} \in X^* \mid 1 + \vec{x} \cdot \vec{y} = 0 \, \forall \vec{y} \in F\} \; . \tag{13}$$

Geometrically, $\widehat{F}$ equals X^* intersected with the dual affine space (11) of the affine span of F. If the origin lies in the interior of the convex body X then $F \mapsto \widehat{F}$ is a one-to-one inclusion-reversing correspondence between the exposed faces of X and of X^* [21]. If X is a tetrahedron, then vertices and faces are exchanged, while edges go to edges.

What we need in order to prove the self-duality of $\mathcal{Q}_N$ is the key fact that a hermitian and unit trace matrix σ is a density matrix if and only if

$$\mathrm{Tr}\, \sigma \rho \; \geq \; 0 \tag{14}$$

for all density matrices ρ. It will be convenient to think of a density matrix ρ as represented by a "vector" u, as in equation (9). As a direct consequence of equation (14) the set of quantum states $\mathcal{Q}_N$ is self-dual in the precise sense that

$$\mathcal{Q}_N - \quad /N = \{u \mid 1/N + \mathrm{Tr}(uv) \geq 0 \, \forall v \in \mathcal{Q}_N - \quad /N\}. \tag{15}$$

In this equation the trace is to be interpreted as a scalar product in a vector space. Duality (13) exchanges faces of rank r (copies of $\mathcal{Q}_r$) and faces of rank $N-r$ (copies of $\mathcal{Q}_{N-r}$).

Self-duality is a key property of state spaces [22, 23], and we will use it extensively when we discuss projections and cross-sections of $\mathcal{Q}_N$. This notion is often introduced in the larger vector space consisting of all hermitian matrices, with the origin at the zero matrix. The set of positive semi-definite matrices forms a cone in this space, with its apex at the origin. It is a cone because any positive semi-definite matrix remains positive semi-definite if multiplied by a positive real number. This defines the rays of the cone, and each ray intersects the set of unit trace matrices exactly once. The dual of this cone is the set of all matrices a such that $\mathrm{Tr}ab \geq 0$ for all matrices b within the cone – and indeed the dual cone is equal to the original, so it is self-dual.

3. An apophatic approach to the qutrit

For $N = 3$ we are dealing with the states of the *qutrit*. The Gell-Mann matrices are a standard choice [16] for the eight matrices γ_i, and the expansion coefficients are $\tau_i = \frac{1}{2}\,\mathrm{Tr}\,\rho\gamma_i$. Unfortunately, although the sufficient conditions for $\vec{\tau}$ to represent a state are known [9, 24, 25], they do not improve much our understanding of the geometry of $\mathcal{Q}_3$.

We know that the set of pure states has 4 real dimensions, and that the faces of $\mathcal{Q}_3$ are copies of the 3D Bloch ball, filling out the seven-dimensional boundary. The centres of these balls touch the largest inscribed sphere of $\mathcal{Q}_3$. But what does it all really look like?

We try to answer this question by presenting some 3D objects, and explaining why they cannot serve as models of $\mathcal{Q}_3$. Apart from the fact that our objects are not eight-dimensional, all of them lack some other features of the set of quantum states.

Figure 3 presents a hairy set which is nice but not convex. Figure 4 shows a ball, and we know that $\mathcal{Q}_3$ is not a ball. It is not a polytope either, so the polytope shown in Figure 5 cannot model the set of quantum states.

Let us have a look at the cylinder shown in Figure 6, and locate the extremal points of the convex body shown. This subset consists of the two circles surrounding both bases. This is a disconnected set, in contrast to the connected set of pure quantum states. However, if one splits the cylinder into two halves and rotates one half by $\pi/2$ as shown in Figure 7, one obtains a body with a connected set of pure states. A similar model can be obtained by taking the convex hull of the seam of a

FIGURE 3. Apophatic approach: this object is *not* a good model of the set $\mathcal{Q}_3$ as it is not a convex set.

FIGURE 4. The set $\mathcal{Q}_3$ is *not* a ball...

FIGURE 5. The set $\mathcal{Q}_3$ is *not* a polytope...

FIGURE 6. The set of pure states in $\mathcal{Q}_3$ is connected, but for the cylinder the pure states form two circles.

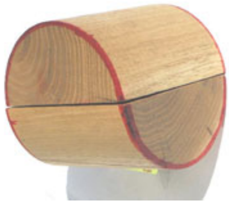


FIGURE 7. This is now the convex hull of a single space curve, but one cannot inscribe copies of the classical set Δ_2 in it.

tennis ball: the one-dimensional seam contains the extremal points of this set and forms a connected set.

Thus the seam of the tennis ball (look again at Figure 4) corresponds to the $4D$ connected set of pure states of $N = 3$ quantum system. The convex hull of the seam forms a $3D$ object which is easy to visualize, and serves as our first rough model of the solid $8D$ body $\mathcal{Q}_3$ of qutrit states. However, a characteristic feature of the latter is that each one of its points belongs to a cross-section which is an equilateral triangle Δ_2. (This is the eigenvector decomposition.) The convex set determined by the seam of the tennis ball, and the set shown in Figure 7, do not have this property.

As we have seen $\mathcal{Q}_3$ can be obtained if we take an equilateral triangle Δ_2 and subject it to $SU(3)$ rotations in eight dimensions. We can try to do something similar in three dimensions. If we rotate a triangle along one of its bisections we obtain a cone, for which the set of extremal states consists of a circle and an apex

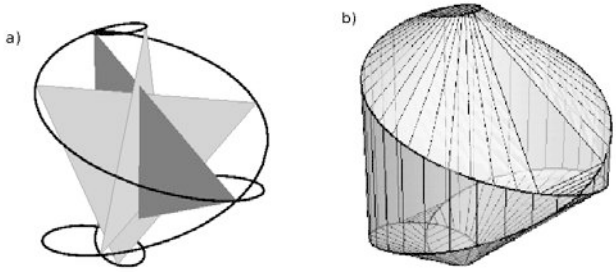


FIGURE 8. a) The space curve $\vec{x}(t)$ modelling pure quantum states is obtained by rotating an equilateral triangle according to equation (16) – three positions of the triangle are shown; b) The convex hull C of the curve models the set of all quantum states.

(see Figure 10 b)), a disconnected set. We obtain a better model if we consider the space curve

$$\vec{x}(t) \;=\; \big(\cos(t)\cos(3t),\; \cos(t)\sin(3t),\; -\sin(t)\big)^{\mathrm{T}} \;. \tag{16}$$

Note that the curve is closed, $\vec{x}(t) = \vec{x}(t+2\pi)$, and belongs to the unit sphere, $||\vec{x}(t)|| = 1$. Moreover

$$||\vec{x}(t) - \vec{x}(t + \tfrac{1}{3}2\pi)|| = \sqrt{3} \tag{17}$$

for every value of t. Hence every point $\vec{x}(t)$ belongs to an equilateral triangle with vertices at

$$\vec{x}(t), \qquad \vec{x}(t+\tfrac{1}{3}2\pi), \qquad \text{and} \qquad \vec{x}(t+\tfrac{2}{3}2\pi)\;.$$

They span a plane including the z-axis for all times t. During the time $\Delta t = \frac{2\pi}{3}$ this plane makes a full turn about the z-axis, while the triangle rotates by the angle $2\pi/3$ within the plane – so the triangle has returned to a congruent position. The curve $\vec{x}(t)$ is shown in Figure 8 a) together with exemplary positions of the rotating triangle, and Figure 8 b) shows its convex hull C. This convex hull is symmetric under reflections in the $(x$-$y)$ and $(x$-$z)$ planes. Since the set of pure states is connected this is our best model so far of the set of quantum pure states, although the likeness is not perfect.

It is interesting to think a bit more about the boundary of C. There are three flat faces, two triangular ones and one rectangular. The remaining part of the boundary consists of ruled surfaces: they are curved, but contain one-dimensional faces (straight lines). The boundary of the set shown in Figure 7 has similar properties. The ruled surfaces of C have an analogue in the boundary of the set of quantum states $\mathcal{Q}_3$, we have already noted that a generic point in the boundary of $\mathcal{Q}_3$ belongs to a copy of $\mathcal{Q}_2$ (the Bloch ball), arising as the intersection of $\mathcal{Q}_3$ with a hyperplane. The flat pieces of C have no analogues in the boundary of $\mathcal{Q}_3$, apart from Bloch balls (rank two) and pure states (rank one) no other faces exist.

Still this model is not perfect: Its set of pure states has self-intersections. Although it is created by rotating a triangle, the triangles are not cross-sections of C. It is not true that every point on the boundary belongs to a face that touches the largest inscribed sphere, as it happens for the set of quantum states [17]. Indeed its boundary is not quite what we want it to be, in particular it has non-exposed faces – a point to which we will return. Above all this is not a self-dual body.

4. A constructive approach

The properties of the eight-dimensional convex set $\mathcal{Q}_3$ might conflict if we try to realize them in dimension three. Instead of looking for an ideal three-dimensional model we shall thus use a complementary approach. To reduce the dimensionality of the problem we investigate cross-sections of the $8D$ set $\mathcal{Q}_3$ with a plane of dimension two or three, as well as its orthogonal projections on these planes – the shadows cast by the body on the planes, when illuminated by a very distant light source. Clearly the cross-sections will always be contained in the projections, but in exceptional cases they may coincide.

What kind of cross-sections arise? In the classical case it is known that every convex polytope arises as a cross-section of a simplex Δ_{N-1} of sufficiently high dimension [21]. It is also true that every convex polytope arises as the projection of a simplex. But what are the cross-sections and the projections of $\mathcal{Q}_N$? There has been considerable progress on this question recently. The convex set is said to be a *spectrahedron* if it is a cross-section of a cone of semi-positive definite matrices of some given size. In the branch of mathematics known as convex algebraic geometry one asks what kind of convex bodies that can be obtained as projections of spectrahedra. Surprisingly, the convex hull of any trigonometric space curve in three dimensions can be so obtained [26]. This includes our set C, which can be shown to be a projection of an eight-dimensional cross-section of the $35D$ set $\mathcal{Q}_6$ of quantum states of size $N = 6$. We do so in the Appendix.

4.1. The duality between projections and cross-sections

In the vector space of traceless hermitian matrices we choose a linear subspace U. The intersection of the convex body $\mathcal{Q}_N$ of quantum states with the subspace $U + \ /N$ through the maximally mixed state $\ /N$ is the cross-section S_U, and the orthogonal projection of $\mathcal{Q}_N$ down to U is the projection P_U. There exists a beautiful relation between projections and cross-sections, holding for self-dual convex bodies such as the classical and the quantum state spaces [14]. For them cross-sections and projections are dual to each other, in the sense that

$$S_U - \ /N = \{u \mid 1/N + \mathrm{Tr}(uv) \geq 0 \ \forall v \in P_U\} \tag{18}$$

and

$$P_U = \{u \mid 1/N + \mathrm{Tr}(uv) \geq 0 \ \forall v \in S_U - \ /N\} \ . \tag{19}$$

This is best explained in a picture (namely Figure 9). A special case of these dualities is the self-duality of the full state-space, equation (15).

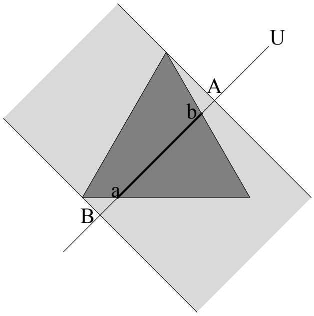


FIGURE 9. The triangle is self-dual. We intersect it with a one-dimensional subspace through the centre, U, and obtain a cross-section extending from a to b. The dual of this line in the plane is a two-dimensional strip, and when we project this onto U we obtain a projection extending from A to B, which is dual to the cross-section within U.

Let us look at two examples for $\mathcal{Q}_3$, choosing the vector space U to be three-dimensional. In Figure 10 a) we show the cross-section containing all states of the form

$$\rho = \begin{pmatrix} 1/3 & x & y \\ x & 1/3 & z \\ y & z & 1/3 \end{pmatrix} , \quad \rho \geq 0 . \tag{20}$$

They form an overfilled tetrapak cartoon [8], also known as an elliptope [27] and an obese tetrahedron [16]. Like the tetrahedron it has six straight edges. Its boundary is known as Cayley's cubic surface, and it is smooth everywhere except at the four vertices. In the picture it is surrounded by its dual projection, which is the convex hull of a quartic surface known as Steiner's Roman surface. To understand the shape of the dual, start with a pair of dual tetrahedra (one of them larger than the other). Then we "inflate" the small tetrahedron a little, so that its facets turn into curved surfaces. It grows larger, so its dual must shrink – the vertices of the dual become smooth, while the facets of the dual will be contained within the original triangles. What we see in Figure 10 a) is a "critical" case, in which the facets of the dual have shrunk to four circular disks that just touch each other in six special points.

In Figure 10 b) we see the cross-section containing all states (positive matrices) of the form

$$\rho = \begin{pmatrix} 1/3 + z/\sqrt{3} & x - \mathrm{i}y & 0 \\ x + \mathrm{i}y & 1/3 + z/\sqrt{3} & 0 \\ 0 & 0 & 1/3 - 2z/\sqrt{3} \end{pmatrix} . \tag{21}$$

This cross-section is a self-dual set, meaning that the projection to this three-dimensional plane coincides with the cross-section. In itself it is the state space of

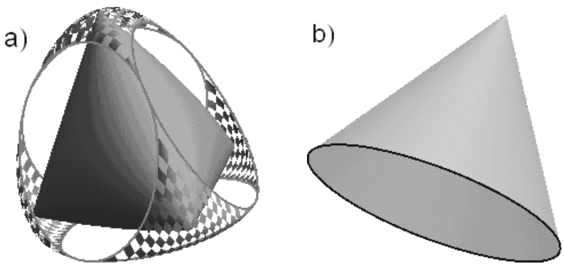


FIGURE 10. a) The cross-section $S_U - \ /3$ defined in (20) of the qutrit quantum states $\mathcal{Q}_3$ is drawn inside the projection P_U of $\mathcal{Q}_3$. b) The cone is self-dual, it is a cross-section and a projection of $\mathcal{Q}_3$ with $S_U - \ /3 = P_U$.

a real subalgebra of the qutrit obervables. There exist also two-dimensional self-dual cross-sections, which are simply copies of the classical simplex Δ_2 – the state space of the subalgebra of diagonal matrices.

4.2. Two-dimensional projections and cross-sections

To appreciate what we see in cross-sections and projections we will concentrate on two-dimensional screens.

We can compute 2D projections using the fact that they are dual to a cross-section. But we can also use the notion of the *numerical range* W of a given operator A, a subset of the complex plane [28, 29, 30]

$$W(A) = \{z \in \mathbb{C} : z = \operatorname{Tr} \rho A,\ \rho \in \mathcal{Q}_N\} \ . \tag{22}$$

If the matrix A is hermitian its numerical range reduces to a line segment, otherwise it is a convex region of the complex plane. To see the connection to projections, observe that changing the trace of A gives rise to a translation of the whole set, so we may as well fix the trace to equal unity. Then we can write for some $\lambda \in$

$$A = \lambda \quad + u + \mathrm{i}v \ , \tag{23}$$

where u and v are traceless hermitian matrices. It follows that the set of all possible numerical ranges $W(A)$ of arbitrary matrices A of order N is affinely equivalent to the set of orthogonal projections of $\mathcal{Q}_N$ on a 2-plane [15, 31]. Thus to understand the structure of projections of $\mathcal{Q}_N$ onto a plane it is sufficient to analyze the geometry of numerical ranges of any operator of size N. For instance, in the simplest case of a matrix A of order $N = 2$, its numerical range forms an elliptical disk, which may reduce to an interval. These are just possible (not necessarily orthogonal) projections of the Bloch ball $\mathcal{Q}_2$ onto a plane.

In the case of a matrix A of order $N = 3$ the shape of its numerical range was characterized algebraically in [32, 33]. Regrouping this classification we divide the possible shapes into four cases according to the number of flat boundary parts:

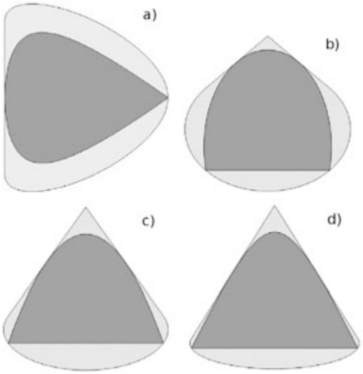


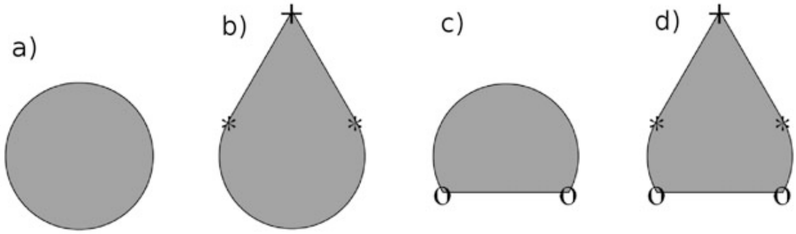
Figure 11. The drawings are dual pairs of planar cross-sections $S_U - \;\, /3$ (dark) and projections P_U (bright) of the convex body of qutrit quantum states $\mathcal{Q}_3$. Drawing a) is obtained from the 3D dual pair in Figure 10 a) and b)–d) are derived from the self-dual cone in Figure 10 b). The cross-sections in b)–d) have an elliptic, parabolic and hyperbolic boundary piece, respectively.

The set W is compact and its boundary ∂W

1. has *no* flat parts. Then W is strictly convex, it is bounded by an ellipse or equals the convex hull of a (irreducible) sextic space curve;
2. has *one* flat part, then W is the convex hull of a quartic space curve – e.g., W is the convex hull of a trigonometric curve known as the cardioid;
3. has *two* flat parts, then W is the convex hull of an ellipse and a point outside it;
4. has *three* flat parts, then W is a triangle with corners at eigenvalues of A.

In case 4 the matrix A is normal, $AA^\dagger = A^\dagger A$, and the numerical range is a projection of the simplex Δ_2 onto a plane. Looking at the planar projections of $\mathcal{Q}_3$ shown in Figure 11 we recognize cases 2 and 3. All four cases are obtained as projections of the Roman surface in Figure 10 a) or the cone shown in Figure 10 b). A rotund shape and one with two flats are obtained as a projection of both 3D bodies. A triangle is obtained from the cone and a shape with one flat from the Roman surface.

In order to actually calculate a $2D$ projection $P := \{(\operatorname{Tr} u\rho, \operatorname{Tr} v\rho)^{\mathrm{T}} \in \;{}^2 \mid \rho \in \mathcal{Q}_3\}$ of the set $\mathcal{Q}_3$ determined by two traceless hermitian matrices u and v one may proceed as follows [28]. For every non-zero matrix F in the real span of u and v we calculate the maximal eigenvalue λ and the corresponding normalized eigenvector $|\psi\rangle$ with $F|\psi\rangle = \lambda|\psi\rangle$. Then $(\langle\psi|u|\psi\rangle, \langle\psi|v|\psi\rangle)^{\mathrm{T}}$ belongs to the projection P, and these points cover all exposed points of P.



Exemplary sets	disk *a*)	drop *b*)	truncated disk *c*)	truncated drop *d*)
non-exposed points $(*)$	no	yes	no	yes
non-polyhedral corners (o)	no	no	yes	yes
set is self-dual	yes	no	no	yes

FIGURE 12. Exemplary convex sets and their duals. Symbols: non-exposed point $(*)$, polyhedral corners $(+)$ and non-polyhedral corners (o). Sets a) and d) are self-dual, while b) and c) is a dual pair. Sets a) and c) have properties like 2D cross-sections of $\mathcal{Q}_N$, while sets a) and b) could be obtained from $\mathcal{Q}_N$ by projection.

4.3. Exposed and non-exposed faces

Cross-sections and projections of the convex body $\mathcal{Q}_N$ of quantum states have a more subtle boundary structure than $\mathcal{Q}_N$ itself.

An exposed face of a convex set X is the intersection of X with an affine hyperplane H such that $X \setminus H$ is convex, i.e., H intersects X only at the boundary. Examples in the plane are the boundary points of the disk in Figure 12 a) or the boundary segments in panels b) and d). A non-exposed face of X is a face of X that is not an exposed face. In dimension two non-exposed faces are non-exposed points, they are the endpoints of boundary segment of X which are not exposed faces by themselves. Examples are the lower endpoints of the boundary segments in Figure 12 b) or d).

It is known that cross-sections of $\mathcal{Q}_N$ have no non-exposed faces. On the other hand the twisted cylinder (see Figure 7) and the convex hull C of the space curve (Figure 8) do have non-exposed faces of dimension one. In contrast to cross-sections, projections of $\mathcal{Q}_N$ can have non-exposed points, see, e.g., the planar projections of $\mathcal{Q}_3$ in Figure 11. They are related to discontinuities in certain entropy functionals (in use as information measures) [34].

The dual concept to exposed face is normal cone [13]. The normal cone of a two-dimensional convex set $X \subset \ ^2$ at $(x_1, x_2)^{\mathrm{T}} \in X$ is

$$\{(y_1, y_2)^{\mathrm{T}} \in \ ^2 \mid (z_1 - x_1)y_1 + (z_2 - x_2)y_2 \leq 0 \ \forall (z_1, z_2)^{\mathrm{T}} \in X\}.$$

The normal cone generalizes outward pointing normal vectors of a smooth boundary curve of X to points $(x_1, x_2)^{\mathrm{T}}$ where this curve is not smooth. Then the dimension of the normal cone is two and we call $(x_1, x_2)^{\mathrm{T}}$ a corner. The examples in Figure 12 have $0, 1, 2, 3$ corners from left to right. There are different types of corners: The top corners of Figure 12 b) and d) are polyhedral, i.e., they are intersections of two boundary segments. If a corner is not the intersection of two boundary segments we call it non-polyhedral. The bottom corners of c) and d) are non-polyhedral corners. Polyhedral and non-polyhedral corners are characterized in [35] in terms of normal cones. From this characterization it follows that any corner of a two-dimensional projection of $\mathcal{Q}_N$ is polyhedral [13]. An analogue property holds in higher dimensions but it cannot be formulated in terms of polyhedra. Figure 11 shows that two-dimensional cross-sections of $\mathcal{Q}_3$ can have non-polyhedral corners.

Given a two-dimensional convex body including the origin in the interior, the duality (13) maps non-exposed points onto the set of non-polyhedral corners of the dual convex body. There will be one or two non-exposed points in each fiber depending on whether the corner does or does not lie on a boundary segment of the dual body [35]. We conclude that a two-dimensional self-dual convex set has no non-exposed points if and only if all its corners are polyhedral.

5. When the dimension matters

So far we have discussed the qutrit, and properties of the qutrit that generalise to any dimension N. But what is special about a quantum system whose Hilbert space has dimension N? The question gains some relevance from recent attempts to find direct experimental signatures of the dimension,

One obvious answer is that if and only if N is a composite number, the system admits a description in terms of entangled subsystems. But we can look for an answer in other directions too. We emphasised that a regular simplex Δ_{N-1} can be inscribed in the quantum state space $\mathcal{Q}_N$. But in the Bloch ball we can clearly inscribe not only Δ_1 (a line segment), but also Δ_2 (a triangle) and Δ_3 (a tetrahedron). If we insist that the vertices of the inscribed simplex should lie on the outsphere of $\mathcal{Q}_N$, and also that the simplex should be centred at the maximally mixed state, then this gives rise to a non-trivial problem once the dimension $N > 2$. This is clear from our model of the latter as the convex hull of the seam of a tennis ball, or in other words because the set of pure states form a very small subset of the outsphere. Still we saw, in Figure 10 a), that not only Δ_2 but also Δ_3 can be inscribed in $\mathcal{Q}_3$, and as a matter of fact so can Δ_5 and Δ_8. But is it always possible to inscribe the regular simplex Δ_{N^2-1} in $\mathcal{Q}_N$, in such a way that the N^2 vertices are pure states? Although the answer is not obvious, it is perhaps surprising to learn that the answer is not known, despite a considerable amount of work in recent years.

The inscribed regular simplices Δ_{N^2-1} are known as symmetric informationally complete positive operator-valued measures, or SIC-POVMs for short. Their existence has been established, by explicit construction, in all dimensions $N \leq 16$ and in a handful of larger dimensions. The conjecture is that they always exist [36]. But the available constructions have so far not revealed any pattern allowing one to write down a solution for all dimensions N. Already here the quantum state space begins to show some N-dependent individuality.

Another question where the dimension matters concerns complementary bases in Hilbert space. As we have seen, given a basis in Hilbert space, there is an $(N-1)$-dimensional cross-section of $\mathcal{Q}_N$ in which these vectors appear as the vertices of a regular simplex Δ_{N-1}. We can – for instance for tomographic reasons [37] – decide to look for two such cross-sections placed in such a way that they are totally orthogonal with respect to the trace inner product. If the two cross-sections are spanned by two regular simplices stemming from two Hilbert space bases $\{|e_i\rangle\}_{i=0}^{N-1}$ and $\{|f_i\rangle\}_{i=0}^{N-1}$, then the requirement on the bases is that

$$|\langle e_i|f_j\rangle|^2 = \frac{1}{N} \tag{24}$$

for all i, j. Such bases are said to be complementary, and form a key element in the Copenhagen interpretation of quantum mechanics [38]. But do they exist for all N?

The answer is yes. To see this, let one basis be the computational one, and let the other be expressed in terms of it as the column vectors of the Fourier matrix

$$F_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^N \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{bmatrix} , \tag{25}$$

where $\omega = e^{2\pi i/N}$ is a primitive root of unity. The Fourier matrix is an example of a *complex Hadamard matrix*, a unitary matrix all of whose matrix elements have the same modulus.

We are interested in finding all possible complementary pairs up to unitary equivalences. The latter are largely fixed by requiring that one member of the pair is the computational basis, since the second member will then be defined by a complex Hadamard matrix. The remaining freedom is taken into account by declaring two complex Hadamard matrices H and H' to be equivalent if they can be related by

$$H' = D_1 P_1 H P_2 D_2 \ , \tag{26}$$

where D_i are diagonal unitary matrices and P_i are permutation matrices.

The task of classifying pairs of cross-sections of $\mathcal{Q}_N$ forming simplices Δ_{N-1} and sitting in totally orthogonal N-planes is therefore equivalent to the problem of classifying complementary pairs of bases in Hilbert space. This problem in turn is equivalent to the problem of classifying complex Hadamard matrices of a given size. But the latter problem has been open since it was first raised by Sylvester and Hadamard, back in the nineteenth century. It has been completely solved only for $N \leq 5$, and it was recently almost completely solved for $N = 6$ [39].

More is known if we restrict ourselves to continuous families of complex Hadamard matrices that include the Fourier matrix. Then it has been known for some time [40] that the dimension of such a family is bounded from above by

$$d_{F_N} = \sum_{k=0}^{N-1} \gcd(k, N) - (2N - 1) \ , \tag{27}$$

where gcd denotes the largest common divisor, and $\gcd(0, N) = N$. We subtracted the $2N-1$ dimensions that arise trivially from equation (26). Moreover, if $N = p^k$ is a power of prime number p this bound is saturated by families that have been constructed explicitly. In particular, if N is a prime number $d_{F_p} = 0$, and the Fourier matrix is an isolated solution. For $N = 4$ on the other hand there exists a one-parameter family of inequivalent complex Hadamard matrices.

Further results on this question were presented in Białowieża [41]. In particular the above bound is not achieved for any N not equal to a prime power and not equal to 6. It turns out that the answer depends critically on the nature of the prime number decomposition of N. Thus, if N is a product of two odd primes the answer will look different from the case when N is twice an odd prime. However, at the moment, the largest non-prime power dimension for which the answer is known – even for this restricted form of the problem – is $N = 12$.

At the moment then, both the SIC problem and the problem of complementary pairs of bases highlight the fact that the choice of Hilbert space dimension N has some dramatic consequences for the geometry of $\mathcal{Q}_N$. Now the basic intuition that drove Mielnik's attempts to generalize quantum mechanics was the feeling that the nature of the physical system should be reflected in the geometry of its convex body of states [1]. Perhaps this intuition will eventually be vindicated within quantum mechanics itself, in such a way that the individuality of the system is expressed in the choice of N?

6. Concluding remarks

As discussed in our work the convex geometry of the set of mixed states of size N is simple for $N = 2$ only and in spite of all our efforts it becomes slightly mysterious already for $N \geq 3$. This observation was also emphasized in a recent paper by Mielnik [42].

Let us try to summarize basic properties of the set $\mathcal{Q}_N$ of mixed quantum states of size $N \geq 3$ analyzed with respect to the flat, Hilbert-Schmidt geometry, induced by the distance (3).

a) The set $\mathcal{Q}_N$ is a convex set of N^2-1 dimensions. It is topologically equivalent to a ball and does not have pieces of lower dimensions ('no hairs').

b) The set $\mathcal{Q}_N$ is inscribed in a sphere of radius $R_N = \sqrt{(N-1)/2N}$, and it contains the maximal ball of radius $r_N = 1/\sqrt{2N(N-1)}$ in the Hilbert-Schmidt distance.
c) The set $\mathcal{Q}_N$ is neither a polytope nor a smooth body.
d) The set of mixed states is self-dual (15).
e) All cross-sections of $\mathcal{Q}_N$ have no non-exposed faces.
f) All corners of two-dimensional projetions of $\mathcal{Q}_N$ are polyhedral.
g) The boundary $\partial\mathcal{Q}_N$ contains all states of less than maximal rank.
h) The set of extremal (pure) states forms a connected $2N-2$-dimensional set, which has zero measure with respect to the N^2-2-dimensional boundary $\partial\mathcal{Q}_N$.
i) Explicit formulae for the volume V and the area A of the $d = N^2-1$-dimensional set $\mathcal{Q}_N$ are known [18]. The ratio Ar/V is equal to the dimension d, which implies that $\mathcal{Q}_N$ has a constant height [17], see (7).

Acknowledgement

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Appendix. Trigonometric curves

We write the convex hull C of the trigonometric space curve in Section 3 as a projection of a cross-section of the 35-dimensional set $\mathcal{Q}_6$ of density matrices. Up to the trace normalization, this problem is solved in [26] for the convex hull of any trigonometric curve $[0, 2\pi) \rightarrow {}^n$. The assumptions are that each of the n coefficient functions of the curve is a trigonometric polynomial of some finite degree $2d$,

$$t \mapsto \textstyle\sum_{k=1}^{d}(\alpha_k \cos(kt) + \beta_k \sin(kt)) + \gamma$$

for real coefficients $\alpha_k, \beta_k, \gamma$.

The space curve (16) lives in dimension $n = 3$, we denote its coefficients by $\vec{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. Using trigonometric formulas and the parametrization $\cos(t) = \frac{y_0^2 - y_1^2}{y_0^2 + y_1^2}$ and $\sin(t) = \frac{2y_0 y_1}{y_0^2 + y_1^2}$ we have

$$\begin{aligned}
1 &\overset{\text{def.}}{=} (y_0^2 + y_1^2)^4 \ , \\
x_1 &= (y_0^2 - y_1^2)^2[(y_0^2 - y_1^2)^2 - 3(2y_0y_1)^2] \ , \\
x_2 &= (y_0^2 - y_1^2)(2y_0y_1)[3(y_0^2 - y_1^2)^2 - (2y_0y_1)^2] \ , \\
x_3 &= -(y_0^2 + y_1^2)^3(2y_0y_1) \ .
\end{aligned}$$

A basis vector of m-variate forms of degree $2d = 8$ is given by

$$\vec{\xi} = (x_0^8, x_0^7 x_1, x_0^6 x_1^2, x_0^5 x_1^3, x_0^4 x_1^4, x_0^3 x_1^5, x_0^2 x_1^6, x_0 x_1^7, x_1^8)^{\mathrm{T}}$$

for the number $m = 1$ used in [26] for the degrees of freedom of the projective coordinates $(y_0 : y_1)$ in the circle ${}^1(\quad))$ and we have

$$(1, x_1, x_2, x_3)^{\mathrm{T}} \; = \; A\vec{\xi}$$

for the 4×9-matrix

$$A \; = \; \begin{pmatrix} 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \\ 1 & 0 & -16 & 0 & 30 & 0 & -16 & 0 & 1 \\ 0 & 6 & 0 & -26 & 0 & 26 & 0 & -6 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix} .$$

Let us denote by $M \succeq 0$ that a complex square matrix M is positive semi-definite. The 5×5 moment matrix of $\vec{u} = (u_1, \ldots, u_9)$ is given by

$$M_4(\vec{u}) \; = \; \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ u_2 & u_3 & u_4 & u_5 & u_6 \\ u_3 & u_4 & u_5 & u_6 & u_7 \\ u_4 & u_5 & u_6 & u_7 & u_8 \\ u_5 & u_6 & u_7 & u_8 & u_9 \end{pmatrix} .$$

Now [26] provides the convex hull representation

$$\begin{aligned} C \; &\overset{\text{def.}}{=} \; \mathrm{conv}\{\vec{x}(t) \in {}^3 \mid t \in [0, 2\pi)\} \\ &= \{\left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right) \in {}^3 \mid \exists \vec{u} \in {}^9 \text{ s.t. } \left(\begin{smallmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right) = A\vec{u} \text{ and } M_4(\vec{u}) \succeq 0\} \end{aligned} \quad (28)$$

which we shall simplify by eliminating the variables $u_1, \ldots, u_4$.

A particular solution of $(1, v_1, v_2, v_3)^{\mathrm{T}} = A\vec{u}$ is

$$\begin{aligned} \widetilde{u}_1 \; &= \; \tfrac{1}{5}(4 + v_1) \; , \quad \widetilde{u}_2 \; = \; \tfrac{1}{44}(3v_2 - 13v_3) \; , \\ \widetilde{u}_3 \; &= \; \tfrac{1}{20}(1 - v_1) \; , \quad \widetilde{u}_4 \; = \; \tfrac{1}{44}(-v_2 - 3v_3) \; , \end{aligned}$$

$\widetilde{u}_5 = \widetilde{u}_6 = \widetilde{u}_7 = \widetilde{u}_8 = \widetilde{u}_9 = 0$. The reduced row echelon form of A being

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 54/5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 39/11 & 0 & 2/11 & 0 \\ 0 & 0 & 1 & 0 & -6/5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2/11 & 0 & 3/11 & 0 \end{pmatrix}$$

and regarding $u_5, \ldots, u_9$ as free variables we have

$$\begin{aligned} u_1 \; &= \; \widetilde{u}_1 - \tfrac{54}{5}u_5 - u_9 \; , & u_2 \; &= \; \widetilde{u}_2 \; - \tfrac{39}{11}u_6 - \tfrac{2}{11}u_8 \; , \\ u_3 \; &= \; \widetilde{u}_3 + \tfrac{6}{5}u_5 - u_7 \; , & u_4 \; &= \; \widetilde{u}_4 + \tfrac{2}{11}u_6 - \tfrac{3}{11}u_8 \; . \end{aligned}$$

One problem remains, the matrix M_4 parametrized by v_1, v_2, v_3 and $u_5, \ldots, u_9$ does not have trace one,

$$\mathrm{Tr}\, M_4 \; = \; u_1 + u_3 + u_5 + u_7 + u_9 \; = \; \tfrac{1}{20}(17 - 172u_5 + 3v_1) \; .$$

This we correct by adding a direct summand to M_4 and by defining

$$M \; = \; \left(\begin{array}{c|c} M_4 & 0 \\ \hline 0 & \frac{172}{20}u_5 + \frac{3}{20}(1 - v_1) \end{array}\right) .$$

If $M_4 \succeq 0$ then $u_5 \geq 0$ follows because u_5 is a diagonal element of M_4 and $-1 \leq v_1 \leq 1$ follows from (28) because $(v_1, v_2, v_3)^{\mathrm{T}} \in C$ is included in the unit ball of ^{3}. This proves $M \succeq 0 \iff M_4 \succeq 0$ and we get

$$C = \{ \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right) \in {}^3 \mid \exists \begin{pmatrix} u_5 \\ \vdots \\ u_9 \end{pmatrix} \in {}^5 \text{ s.t. } M \succeq 0 \} \,.$$

We conclude that C is a projection of the eight-dimensional spectrahedron

$$\{(v_1, v_2, v_2, u_5, \ldots, u_9)^{\mathrm{T}} \in {}^{3+5} \mid M \succeq 0\},$$

which is a cross-section of $\mathcal{Q}_6$.

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Solution Hierarchies for the Painlevé IV Equation

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To Professor Bogdan Mielnik on his 75th Birthday

Abstract. We will obtain real and complex solutions of the Painlevé IV equation through supersymmetric quantum mechanics. The real solutions will be classified into several hierarchies, and a similar procedure will be followed for the complex solutions.

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Keywords. Factorization method, supersymmetric quantum mechanics, Painlevé equations.

1. Introduction

The Painlevé equations can be seen as the nonlinear analogues of the classical linear equations associated to the well-known special functions [1, 2]. They have been identified as the most important non-linear ordinary differential equations [3]. Although discovered by strictly mathematical considerations, nowadays they are widely used to describe several physical phenomena [4]. In particular, the Painlevé IV equation (P_{IV}) is relevant in fluid mechanics, non-linear optics, and quantum gravity [5].

On the other hand, since its birth supersymmetric quantum mechanics (SUSY QM) catalyzed the study of exactly solvable Hamiltonians and gave a new insight into the algebraic structure characterizing these systems. Historically, the essence of SUSY QM was developed first as Darboux transformation in mathematical physics [6] and as factorization method in quantum mechanics [7, 8]. Moreover, through SUSY QM one can obtain quantum systems described by second-order polynomial Heisenberg algebras (PHA), whose Hamiltonians have the standard Schrödinger form and their differential ladder operators are of third order. It has been shown that there is a connection between these systems and solutions $g(x;a,b)$ of P_{IV} [2].

The P_{IV} solutions can be grouped into several hierarchies, according to the family of special functions they are related with. This classification can be easily done for the class of real solutions [9], but it can be as well performed for the recently found complex solutions [10], which is our aim here. To do that, we have arranged this paper as follows: in Section 2 we shall present the general framework of SUSY QM and PHA. In the next section we will generate the real and complex solutions to P_{IV}; then, in Section 4 we will study the real solution hierarchies and we shall analyze the domain of the parameter space (a, b) where they are to be found. In Section 5 we do the same for the complex solution. We present our conclusions in Section 6.

2. General framework of SUSY QM and PHA

In the kth order SUSY QM one starts from a given solvable Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \tag{1}$$

and generates a chain of first-order intertwining relations [11, 12, 13]

$$H_j A_j^+ = A_j^+ H_{j-1}, \quad H_{j-1} A_j^- = A_j^- H_j, \tag{2}$$

$$H_j = -\frac{1}{2}\frac{d^2}{dx^2} + V_j(x), \quad A_j^\pm = \frac{1}{\sqrt{2}}\left[\mp\frac{d}{dx} + \alpha_j(x,\epsilon_j)\right], \quad j = 1,\dots,k. \tag{3}$$

By plugging equations (3) into equation (2) we obtain

$$\alpha_j'(x,\epsilon_j) + \alpha_j^2(x,\epsilon_j) = 2[V_{j-1}(x) - \epsilon_j], \quad V_j(x) = V_{j-1}(x) - \alpha_j'(x,\epsilon_j). \tag{4}$$

We are interested in the final Riccati solution $\alpha_k(x,\epsilon_k)$, which turns out to be determined either by k solutions $\alpha_1(x,\epsilon_j)$ of the initial Riccati equation

$$\alpha_1'(x,\epsilon_j) + \alpha_1^2(x,\epsilon_j) = 2[V_0(x) - \epsilon_j], \quad j = 1,\dots,k, \tag{5}$$

or by k solutions $u_j \propto \exp(\int \alpha_1(x,\epsilon_j)dx)$ of the associated Schrödinger equation

$$H_0 u_j = -\frac{1}{2}u_j'' + V_0(x)u_j = \epsilon_j u_j, \quad j = 1,\dots,k. \tag{6}$$

Thus, there is a pair of kth order operators interwining the initial H_0 and final Hamiltonians H_k, namely,

$$H_k B_k^+ = B_k^+ H_0,\ H_0 B_k^- = B_k^- H_k,\ B_k^+ = A_k^+ \dots A_1^+,\ B_k^- = A_1^- \dots A_k^-. \tag{7}$$

The normalized eigenfunctions $\psi_n^{(k)}$ of H_k, associated to the eigenvalues E_n, and the k additional eigenstates $\psi_{\epsilon_j}^{(k)}$ associated to the eigenvalues ϵ_j which are annihilated by B_k^- $(j = 1,\dots,k)$, are given by [9, 14]:

$$\psi_n^{(k)} = \frac{B_k^+\psi_n}{\sqrt{(E_n-\epsilon_1)\dots(E_n-\epsilon_k)}},\ \psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1,\dots,u_{j-1},u_{j+1},\dots,u_k)}{W(u_1,\dots,u_k)}. \tag{8}$$

Note that, in this formalism the obvious restriction $\epsilon_j < E_0 = 1/2$ naturally arises if we want to avoid singularities in $V_k(x)$.

On the other hand, a mth order PHA is a deformation of the Heisenberg-Weyl algebra of kind [14, 15, 16]:

$$[H, L^{\pm}] = \pm L^{\pm}, \quad [L^-, L^+] \equiv Q_{m+1}(H+1) - Q_{m+1}(H) = P_m(H), \tag{9}$$

$$Q_{m+1}(H) = L^+L^- = (H - \mathcal{E}_1)\dots(H - \mathcal{E}_{m+1}), \tag{10}$$

where $P_m(x)$ is a polynomial of order m in x and $\mathcal{E}_i$ are the zeros of $Q_{m+1}(H)$, which correspond to the energies associated to the extremal states of H.

Now, in the differential representation of the second-order PHA ($m = 2$), L^+ is a third-order differential ladder operator, chosen by simplicity as [17]:

$$L^+ = L_1^+ L_2^+, \; L_1^+ = \frac{1}{\sqrt{2}}\left[-\frac{d}{dx} + f(x)\right], \; L_2^+ = \frac{1}{2}\left[\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x)\right]. \tag{11}$$

These operators satisfy the following relationships:

$$HL_1^+ = L_1^+(H_{\rm a} + 1), \quad H_{\rm a}L_2^+ = L_2^+H \quad \Rightarrow \quad [H, L^+] = L^+, \tag{12}$$

$H_{\rm a}$ being an auxiliary Schrödinger Hamiltonian. Using the standard first and second-order SUSY QM one obtains

$$f = x + g, \qquad h = -x^2 + \frac{g'}{2} - \frac{g^2}{2} - 2xg + a, \tag{13}$$

$$V = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \mathcal{E}_1 - \frac{1}{2}, \tag{14}$$

$$g'' = \frac{g'^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2\left(x^2 - a\right)g + \frac{b}{g}. \tag{15}$$

The last one is the Painlevé IV equation (P_{IV}) with parameters

$$a = \mathcal{E}_2 + \mathcal{E}_3 - 2\mathcal{E}_1 - 1, \quad b = -2(\mathcal{E}_2 - \mathcal{E}_3)^2. \tag{16}$$

If the $\mathcal{E}_i, \; i = 1, 2, 3$ are real, we will obtain real parameters a, b for equation (15).

3. Real and complex solutions of P_{IV} with real parameters

It is well known that the first-order SUSY partner Hamiltonians of the harmonic oscillator are naturally described by second-order PHA, which are connected with P_{IV}. Furthermore, there is a theorem stating the conditions for the hermitian higher-order SUSY partner Hamiltonians of the harmonic oscillator to have this kind of algebras (see [9]). The main requirement is that the k Schrödinger seed solutions have to be connected in the way

$$u_j = (a^-)^{j-1}u_1, \qquad \epsilon_j = \epsilon_1 - (j-1), \qquad j = 1, \dots, k, \tag{17}$$

where a^- is the standard annihilation operator of H_0 so that u_1 is the only free seed.

If u_1 is a real solution of equation (6) without zeros, associated to a real factorization energy ϵ_1 such that $\epsilon_1 < E_0 = 1/2$, then all u_j are also real and, consequently, the solutions to P_{IV} are also real. On the other hand, if we use the formalism as in [9] with $\epsilon_1 > E_0$, we would obtain only singular SUSY transformations. In order to avoid this we will instead employ complex SUSY transformations. The simplest way to implement them is to use a complex linear combination of the two standard linearly independent real solutions which, up to an unessential factor, leads to the following complex solutions depending on a complex constant $\Lambda = \lambda + i\kappa$ $(\lambda, \kappa \in \mathbb{R})$ [18]:

$$u(x;\epsilon) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-2\epsilon}{4}, \frac{1}{2}; x^2\right) + x\,\Lambda\,{}_1F_1\left(\frac{3-2\epsilon}{4}, \frac{3}{2}; x^2\right)\right], \tag{18}$$

where ${}_1F_1$ is the *confluent hypergeometric function.* The results for the real case [19] are obtained by making $\kappa = 0$ and expressing $\Lambda = \lambda$, with $\nu \in \mathbb{R}$, as

$$\Lambda = \lambda = 2\nu \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}. \tag{19}$$

Note that the extremal states of H_k and their corresponding energies are given by

$$\psi_{\mathcal{E}_1} \propto \frac{W(u_1,\ldots,u_{k-1})}{W(u_1,\ldots,u_k)}, \quad \mathcal{E}_1 = \epsilon_k = \epsilon_1 - (k-1), \tag{20}$$

$$\psi_{\mathcal{E}_2} \propto B_k^+ e^{-x^2/2}, \qquad \mathcal{E}_2 = \frac{1}{2}, \tag{21}$$

$$\psi_{\mathcal{E}_3} \propto B_k^+ a^+ u_1, \qquad \mathcal{E}_3 = \epsilon_1 + 1. \tag{22}$$

Recall that all the u_j satisfy equation (17) and u_1 corresponds to the general solution given in equation (18).

Hence, through this formalism we will obtain a kth order SUSY partner potential $V_k(x)$ of the harmonic oscillator and a P_{IV} solution $g_k(x;\epsilon_1)$, both of which can be chosen real or complex, in the way

$$V_k(x) = \frac{x^2}{2} - \{\ln[W(u_1,\ldots,u_k)]\}'', \tag{23}$$

$$g_k(x;\epsilon_1) = -x - \{\ln[\psi_{\mathcal{E}_1}(x)]\}'. \tag{24}$$

For $k = 1$, the first-order SUSY transformation and equation (24) lead to what is known as *one-parameter solutions* to P_{IV}, due to the restrictions imposed by equation (16) onto the parameters a, b of P_{IV} which make them both depend on ϵ_1 [20]. For this reason, this family of solutions cannot be found in any point of the parameter space (a, b), but only in the subspace defined by the curve $\{(a(\epsilon_1), b(\epsilon_1)),\ \epsilon_1 \in \mathbb{R}\}$ consistent with equations (16). Then, by increasing the order of the transformation to an arbitrary integer k, we will expand this subspace for obtaining k different families of one-parameter solutions. This procedure is analogous to iterated auto-Bäcklund transformations [21]. Note also that by making cyclic permutations of the indices of the three energies $\mathcal{E}_i$ and the corresponding extremal states of equations (20)–(22) (when they have no nodes), we expand the solution families to three different sets, defined by

$$a_1 = -\epsilon_1 + 2k - \frac{3}{2}, \quad b_1 = -2\left(\epsilon_1 + \frac{1}{2}\right)^2, \tag{25}$$

$$a_2 = 2\epsilon_1 - k, \qquad b_2 = -2k^2, \tag{26}$$

$$a_3 = -\epsilon_1 - k - \frac{3}{2}, \quad b_3 = -2\left(\epsilon_1 - k + \frac{1}{2}\right)^2, \tag{27}$$

where we have added an index corresponding to the extremal state given by equations (20)–(22). Therefore we obtain three different solution families of P_{IV} through equations (18)–(24). The first family includes non-singular real and complex solutions, while the second and third ones can give just non-singular strictly complex solutions, with singularities appearing in the real case.

4. Real solution hierarchies

The solutions $g_k(x;\epsilon_1)$ of the Painlevé IV equation can be classified according to the explicit functions on which they depend [20]. In the real case, see equations (18) and (24) with the condition given in equation (19), the solutions are expressed in terms of the confluent hypergeometric function ${}_1F_1$, although for specific values of the parameter ϵ_1 they can be reduced to the *error function* $\text{erf}(x)$. Moreover, for particular parameters ϵ_1 and ν_1, they simplify further to rational solutions.

Let us remark that we are interested in non-singular SUSY partner potentials and the corresponding non-singular solutions of P_{IV}. Note that the same set of real solutions to P_{IV} can be obtained through inverse scattering techniques [4] (compare the solutions of [20] with those of [9]).

4.1. Confluent hypergeometric function hierarchy

In general, the solutions of P_{IV} are expressed in terms of two confluent hypergeometric functions. For example, let us write down the explicit formula for $g_1(x;\epsilon_1)$ in terms of the parameters ϵ_1, ν_1 (with $\epsilon_1 < 1/2$ and $|\nu_1| < 1$ to avoid singularities):

$$\begin{aligned} g_1(x,\epsilon_1) &= \frac{2\nu_1\Gamma\left(\frac{3-2\epsilon_1}{4}\right)\left[3\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2};x^2\right) - (2\epsilon_1+3)x^2\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{5}{2};x^2\right)\right]}{3\Gamma\left(\frac{1-2\epsilon_1}{4}\right)\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{1}{2};x^2\right) + 6\nu_1 x\Gamma\left(\frac{3-2\epsilon_1}{4}\right)\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2};x^2\right)} \\ &+ \frac{3x(2\epsilon_1+1)\Gamma\left(\frac{1-2\epsilon_1}{4}\right)\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{3}{2};x^2\right)}{3\Gamma\left(\frac{1-2\epsilon_1}{4}\right)\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{1}{2};x^2\right) + 6\nu_1 x\Gamma\left(\frac{3-2\epsilon_1}{4}\right)\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2};x^2\right)}. \end{aligned} \tag{28}$$

The explicit analytic formulas for higher-order solutions $g_k(x;\epsilon_1)$ can be obtained from expression (24), and they have a similar form as in equation (28).

4.2. Error function hierarchy

It is interesting to analyze the possibility of reducing the explicit form of the P_{IV} solution to the error function. To do that, let us fix the factorization energy in such a way that any of the two hypergeometric series of equation (18) reduces to that function. This can be achieved for $\epsilon_1 = -(2m+1)/2$, with $m \in \mathbb{N}$. By defining $\varphi_{\nu_1}(x) \equiv \sqrt{\pi} e^{x^2}[1 + \nu_1 \operatorname{erf}(x)]$, we can write down simple expressions for $g_k(x, \epsilon_1)$ for some specific parameters k and ϵ_1:

$$g_1(x; -5/2) = \frac{4[\nu_1 + x\varphi_{\nu_1}(x)]}{2\nu_1 x + (1 + 2x^2)\varphi_{\nu_1}(x)}, \tag{29}$$

$$g_2(x; -1/2) = \frac{4\nu_1[\nu_1 + 6x\varphi_{\nu_1}(x)]}{\varphi_{\nu_1}(x)[\varphi_{\nu_1}^2(x) - 2\nu_1 x\varphi_{\nu_1}(x) - 2\nu_1^2]}. \tag{30}$$

4.3. Rational hierarchy

Now, let us look for the restrictions needed to reduce the explicit form of equation (24) to non-singular rational solutions. To achieve this, once again the factorization energy ϵ_1 has to be a negative half-integer, but depending on the ϵ_1 taken, just one of the two hypergeometric functions is reduced to a polynomial. Thus, we need to choose additionally the parameter $\nu_1 = 0$ or $\nu_1 \to \infty$ to keep the appropriate hypergeometric function. However, u_1 have a zero at $x = 0$ when $\nu_1 \to \infty$, which will produce one singularity for the corresponding P_{IV} solution. Hence, we should take $\nu_1 = 0$ and $\epsilon_1 = -(4m + 1)/2$ with $m \in \mathbb{N}$. Departing from Schrödinger solutions with these ν_1, ϵ_1 we get some explicit expressions for the $g_k(x; \epsilon_1)$ of the rational hierarchy:

$$g_1(x; -5/2) = \frac{4x}{1 + 2x^2}, \tag{31}$$

$$g_2(x; -5/2) = -\frac{4x}{1 + 2x^2} + \frac{16x^3}{3 + 4x^4}, \tag{32}$$

$$g_3(x; -5/2) = -\frac{16x^3}{3 + 4x^4} + \frac{12(3x + 4x^3 + 4x^5)}{9 + 18x^2 - 12x^4 + 8x^6}, \tag{33}$$

which are plotted in Figure 1.

4.4. First kind modified Bessel function hierarchy

Another interesting case associated to a special function arises for $\epsilon_1 = -m$, $m \in \mathbb{N}$, which leads to the modified Bessel function of first kind. We write down an example of one solution belonging to such a hierarchy:

$$g_1(x; 0) = \frac{\nu_1(1 - x^2)I_{\frac{1}{4}}\left(\frac{x^2}{2}\right) + x^2\left[-I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) + I_{\frac{3}{4}}\left(\frac{x^2}{2}\right) + \nu_1 I_{\frac{5}{4}}\left(\frac{x^2}{2}\right)\right]}{x\left[I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) + \nu_1 I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right]}. \tag{34}$$

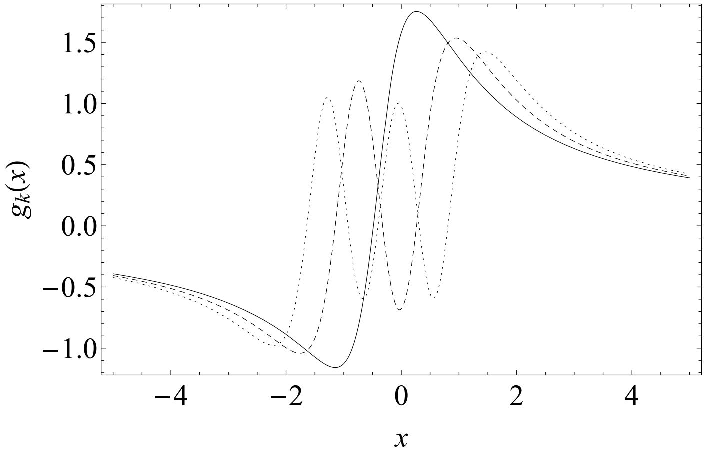

FIGURE 1. The P_{IV} solutions given by equations (31)–(33).

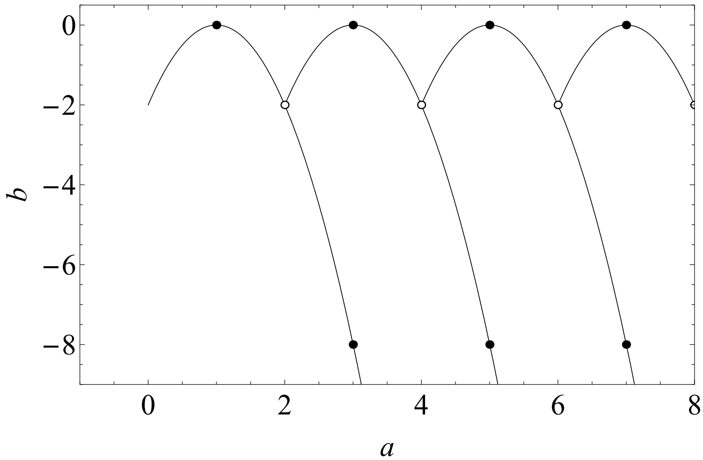

FIGURE 2. Parameter space for real P_{IV} solutions. The lines represent solutions of the confluent hypergeometric function hierarchy, the black dots of the error function hierarchy, and the white dots of the rational and error function hierarchies.

5. Complex solution hierarchies

Let us study the complex solutions subspace, i.e., we use the complex linear combination of equation (18) and the associated P_{IV} solution of equation (24). This allows the use of seeds u_1 with $\epsilon_1 \geq 1/2$ but without producing singularities. Moreover, the complex case is richer than the real one, since all three extremal states of equations (20)–(22) lead to non-singular complex P_{IV} solution families.

5.1. Confluent hypergeometric hierarchy

As in the real case, in general the solutions of P_{IV} are expressed in terms of two confluent hypergeometric functions. In particular, the explicit formula for the first family $g_1(x;\epsilon_1)$ in terms of the parameters ϵ_1, Λ is given by

$$g_1(x,\epsilon_1) = \frac{\Lambda\left[3\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2};x^2\right) - (2\epsilon_1+3)x^2\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{5}{2};x^2\right)\right]}{3\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{1}{2},x^2\right)+\Lambda\, x\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2},x^2\right)} - \frac{3x(2\epsilon_1+1)\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{3}{2};x^2\right)}{3\,{}_1F_1\left(\frac{1-2\epsilon_1}{4},\frac{1}{2},x^2\right)+\Lambda\, x\,{}_1F_1\left(\frac{3-2\epsilon_1}{4},\frac{3}{2},x^2\right)}. \quad (35)$$

Once again, for all families the explicit analytic formulas for the higher-order solutions $g_k(x;\epsilon_1)$ can be obtained through the formula (24).

5.2. Error function hierarchy

If we choose the parameter $\epsilon_1 = -(2m+1)/2$ with $m \in \mathbb{N}$, as in the real case, we obtain the error function hierarchy. In terms of the auxiliary function $\phi_\Lambda = \mathrm{e}^{x^2}[4+\Lambda\pi^{1/2}\mathrm{erf}(x)]$, a solution from the third family is written as:

$$g_1(x;-5/2) = \frac{4\Lambda + 4x\phi_\Lambda(x)}{2\Lambda x + (1+2x^2)\phi_\Lambda(x)}. \quad (36)$$

5.3. Imaginary error function hierarchy

Different to the real case, now we can use $\epsilon_1 \geq 1/2$, giving place to more solution families. This is clear by comparing the real and complex parameter spaces of solutions from Figure 2 and Figure 3. By defining a new auxiliary function $\phi i_\Lambda = \mathrm{e}^{-x^2}[4+\Lambda\pi^{1/2}\mathrm{erfi}(x)]$, where $\mathrm{erfi}(x)$ is the *imaginary error function*, we can write down an explicit solution from the third family

$$g_1(x;5/2) = \frac{4\Lambda(1-x^2) + 2x(-3+2x^2)\phi i_\Lambda(x)}{2\Lambda x + (1-2x^2)\phi i_\Lambda(x)}. \quad (37)$$

5.4. First kind modified Bessel function hierarchy

Let us write down an example of the solution of this hierarchy for $\lambda = 0$, $\kappa = 1$, $\Lambda = i$, i.e., u_1 is a purely imaginary linear combination of the two standard real

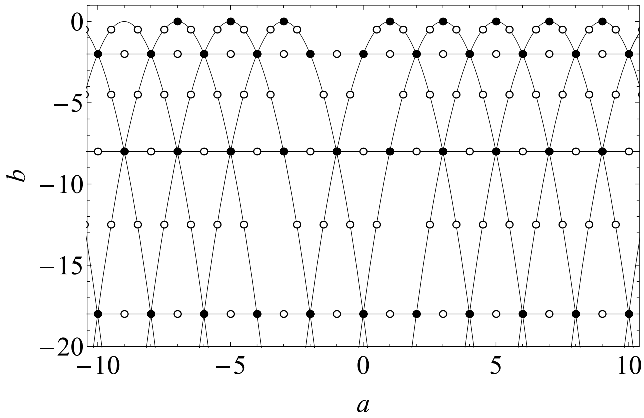

FIGURE 3. Parameter space for complex solution hierarchies. The lines correspond to the confluent hypergeometric function, the black dots to the error function or the imaginary error function, and the white dots to the first kind modified Bessel function.

solutions associated to $\epsilon_1 = 0$:

$$g_1(x;0) = \frac{x\Gamma\left(\frac{3}{4}\right)\left[I_{\frac{3}{4}}\left(\frac{x^2}{2}\right) - I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right] + 2ix\Gamma\left(\frac{5}{4}\right)\left[I_{-\frac{3}{4}}\left(\frac{x^2}{2}\right) - I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right]}{\Gamma\left(\frac{3}{4}\right)I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) + 2i\Gamma\left(\frac{5}{4}\right)I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)}. \tag{38}$$

Its real and imaginary parts are plotted in Figure 4.

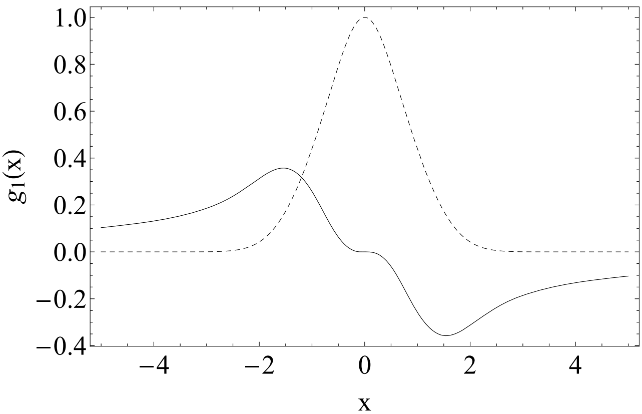

FIGURE 4. Real (solid curve) and imaginary (dashed curve) parts of a complex solution to P_{IV}. The plot corresponds to $k = 1$, $\epsilon_1 = 0$, $\lambda = 0$, and $\kappa = 1$.

6. Conclusions

In this paper we have discussed a general method to obtain real and complex solutions of Painlevé IV equation by using SUSY QM, which is closely related to the factorization method. Through this scheme we have shown that real factorization energies can be used to obtain P_{IV} solutions with real parameters a, b. We have shown the existence of more solutions in the complex case than in the real one by studying in detail the parameter space (a, b).

We have classified the solutions into hierarchies arising both in the real and in the complex cases. Both classifications became very similar, except for a hierarchy which cannot be obtained in the real case. A further study of the Painlevé IV equation with complex parameters is currently under development.

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The Marvelous Consequences of Hardy Spaces in Quantum Physics

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Abstract. Dynamical differential equations, like the Schrödinger equation for the states, or the Heisenberg equation for the observables, need to be solved under boundary conditions. The original boundary condition of von Neumann, the Hilbert space axiom, required that the allowed wave functions are Lebesgue square integrable. This leads by a mathematical theorem of Stone-von Neumann to the unitary group evolution meaning the time t extends over $-\infty < t < +\infty$. Physicists do not use Lebesgue integrals but followed a different path using almost exclusively the Dirac formalism and well-behaved (Schwartz) functions. This led the mathematicians to Schwartz-Rigged Hilbert spaces (Gelfand triplets), which are the mathematical core of Dirac's bra-ket formalism. This is insufficient for a theory that includes resonance and decay phenomena, which requires analytic continuation in energy E in order to accommodate exponentially decaying Gamow kets, Breit-Wigner (Lorentzian) resonances, and Lippmann-Schwinger kets. This leads to a pair of Rigged Hilbert Spaces of smooth Hardy functions, one representing the prepared states of scattering experiments (preparation apparatus) and the other representing detected observables (registration apparatus). A mathematical consequence of the Hardy space axiom is that the time evolution is asymmetric given by the semi-group, i.e., $t_0 \leq t < +\infty$, with a *finite* t_0. What would the meaning of that t_0 be?

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1. Introduction

The fundamental idea of quantum physics is the division of an experiment into the preparation of a state, represented by a self-adjoint state operator ρ or W (or by a vector ϕ if $W = |\phi\rangle\langle\phi|$), and the registration of an observable, represented by self-adjoint operators A, B, ..., (or in a special case represented by a vector ψ if the observable is given by $A = |\psi\rangle\langle\psi|$).

The experimental quantities are the Born probabilities $\mathcal{P}_W(A(t))$ to measure (or "register") the observable A in the state W. They are measured in experiments as ratios of large numbers $\frac{N(t)}{N}$ of detector counts and calculated in quantum theory as the Born Probabilities

$$\mathcal{P}_W(A(t)) = \mathrm{Tr}(W_0\, A(t)) = \mathrm{Tr}(W(t)\, A) \tag{1}$$

In the special case of

$$A(t) = |\psi(t)\rangle\langle\psi(t)| \qquad W = |\phi\rangle\langle\phi| \qquad \text{(Heisenberg picture)}$$

or in the case of

$$A = |\psi\rangle\langle\psi| \qquad W = |\phi(t)\rangle\langle\phi(t)| \qquad \text{(Schrödinger picture)}\,,$$

these probabilities are given by

$$\mathcal{P}_\phi(|\psi(t)\rangle) = |\langle\psi\,|\phi(t)\rangle|^2 = |\langle\psi(t)\,|\phi\rangle|^2 \ . \tag{2}$$

The comparison between $\mathcal{P}_W(A(t))$ which is calculated in theory, and $\frac{N(t)}{N}$, which are observed in the experiment by detector counts, tests the agreement between theory and experiment:

$$\mathcal{P}_W(A(t)) \simeq \frac{N(t)}{N}\,. \tag{3}$$

The probabilities of the observable $A(t)$ in a state W are thus compared with the experimental value $\frac{N(t)}{N}$.[1]

In a scattering experiment, for example, the preparation consists of the acceleration and the collimation of the projectile, which interacts with the target, perhaps forming a resonance. The registration consists of the detection of scattered particles, e.g., the decay products of the resonance which, e.g., decays into different channels characterized by A. To distinguish what is prepared in the preparation process from what is detected in the registration process, one uses different words: state for what is prepared and observable for what is detected or registered (counted by a detector). Despite this experimental distinction between prepared state and detected observable, conventional quantum mechanics does usually not distinguish in the mathematical description between a state and an observable. For instance, a pure state is represented by the projection operator $|\phi\rangle\langle\phi|$ with $\phi \in \mathcal{H}$, the Hilbert space. But any $\phi \in \mathcal{H}$ could as well represent an observable $|\phi\rangle\langle\phi|$.

Thus under the conventional, orthodox axioms of quantum theory, any vector $\phi \in \mathcal{H}$ can represent a state, but it could as well represent an observable $|\phi\rangle\langle\phi|$. And any $|\phi\rangle\langle\phi|$ or ϕ can represent an observable but it could as well represent a state. In contrast to this:

Experimentally, an observable is defined by a registration apparatus (e.g., a detector or counter) and a state is defined by a preparation apparatus (e.g., accelerator). Thus, observables and states are different physical concepts. Therefore,

[1]The sign $\simeq$ indicates that this comparison between the continuous function of t calculated in the theory on the l.h.s of (3) and the rational numbers on the r.h.s of (3) of counting rate can in principle only be confirmed to a certain level of accuracy.

they should also be distinguished in their mathematical description. For instance, would it not be better if the set of observables and the set of states came from different subspace of the Hilbert space? It could even be the case that these different subspaces are "dense" in the same Hilbert space.[2]

2. The mathematics of time symmetric quantum mechanics and its conflict with causality

2.1. The Hilbert space boundary condition of the dynamical equations

Time evolution in quantum mechanics is described in various ways, called pictures. In the Schrödinger picture, the time evolution is described as the evolution of the state vector $\phi(t)$ (or of the state operator $W(t)$ also called density operator with the property $W(t) = W^\dagger(t)$, Tr $W(t) = 1$). The dynamical equation is the Schrödinger equation for the state vector $\phi(t)$

$$i\hbar\frac{\partial}{\partial t}\phi(t) = H\phi(t)\,. \tag{4a}$$

The Hamiltonian H is a self-adjoint or essentially self-adjoint operator; it represents the energy operator or Hamiltonian of the quantum mechanical system.

The dynamical equation for the statistical operator $W(t)$ is the von-Neumann equation

$$i\hbar\frac{\partial}{\partial t}W(t) = [H, W(t)]\,, \tag{4b}$$

which leads to (4a) in case that $W(t) = |\phi(t)\rangle\langle\phi(t)|$ is a pure state. In the Heisenberg picture, the dynamics is described by the Heisenberg equation for the observables represented by a hermitian operator $\Lambda(t)$ ($\Lambda^\dagger = \Lambda$)

$$i\hbar\frac{\partial}{\partial t}\Lambda(t) = -[H, \Lambda(t)]\,. \tag{5a}$$

If the observable is the special "property" $\Lambda = |\psi\rangle\langle\psi|$, the time evolution of the Heisenberg equation for this "observable vector" $\psi(t)$ is

$$i\hbar\frac{\partial}{\partial t}\psi(t) = -H\psi(t)\,. \tag{5b}$$

To solve these differential equations (the Heisenberg or Schrödinger equations[3]), one needs to impose "boundary conditions". The boundary conditions specify the set of vectors $\{\phi(t)\}$ or $\{\psi(t)\}$ that are solutions of the differential equation (4a) or (5b). The original boundary condition introduced by von-Neumann is the "Hilbert space boundary condition":

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[2]This will turn out to be the case for the subspace of detected out-observables Φ_+ and the subspace of prepared in-states Φ_-, which we shall introduce below.

[3]Usually one calls (4a) the Schrödinger equation and (5a) the Heisenberg equation; (4a) is the special case $W(t) = |\phi(t)\rangle\langle\phi(t)|$ of (4b) and (5b) is the special case $A(t) = |\psi(t)\rangle\langle\psi(t)|$ of (5a).

Find

$$\begin{array}{cc} \text{Set of states:} & \text{with } \phi \in \text{Hilbert space}\, \mathcal{H}\,, \\ \text{(all possible solutions of (4a))} & \end{array} \tag{6a}$$

and

$$\begin{array}{cc} \text{Set of observables:} & \text{with } \psi \in \text{Hilbert space}\, \mathcal{H}\,. \\ \text{(all possible solutions of (5b))} & \end{array} \tag{6b}$$

It follows from a theorem of Stone and von Neumann [1] that the solutions of the Schrödinger equation (4a) under this boundary condition (6a) are:

$$\phi(t) = U^\dagger(t)\phi = \mathrm{e}^{-iHt/\hbar}\phi \qquad -\infty < t < +\infty \tag{7a}$$

where $\phi = \phi(t = 0)$.[4]

Similarly by the same theorem follows that all solutions of Heisenberg equation (5b) for observable vector ψ under the condition (6b) are given by

$$\psi(t) = U(t)\psi = \mathrm{e}^{iHt/\hbar}\psi, \qquad \text{with } -\infty < t < +\infty. \tag{7b}$$

where $\psi = \psi(t = 0)$.
Equations (7) describe the unitary group evolution given by the unitary operator $U^\dagger(t) = \mathrm{e}^{-iHt/\hbar}$, or by $U(t) = \mathrm{e}^{iHt/\hbar}$. These operators form a one-parameter group of unitary operators: $U^\dagger(t) = U(-t) = U^{-1}(t)$.

The solutions of (5a) and (4b) are also given by the unitary group:

$$\Lambda(t) = \mathrm{e}^{iHt/\hbar}\Lambda_0\, \mathrm{e}^{-iHt/\hbar}\,, \qquad -\infty < t < +\infty \tag{8a}$$

and

$$W(t) = \mathrm{e}^{-iHt/\hbar}W_0\, \mathrm{e}^{iHt/\hbar}\,, \qquad -\infty < t < +\infty\,. \tag{8b}$$

Here Λ_0 and W_0 are the observable Λ and density operator W at a time t_0 (any **finite** time, e.g., $t_0 = 0$ as chosen in (8)).

The Hilbert space $\mathcal{H}$ is a linear scalar product space in which the scalar products are defined by *Lebesgue* integrals

$$(\psi|\phi) = \int_{0\ \text{Lebesgue}}^{\infty} dE\ \overline{\psi(E)}\ \phi(E) \tag{9a}$$

Here we have chosen the energy wave functions $\psi(E)$ and $\phi(E)$, but the same kind of integration is assumed also for the position wave functions $\phi(x)$, the momentum wave functions $\phi(p)$ and the function of any continuous variable.

The Hilbert space $\mathcal{H}$ is a complete space, this means all Cauchy sequences in $\mathcal{H}$ have a limit point in this space $\mathcal{H}$. The convergence is defined with respect to the norm defined in (10) below. However, in order that $\mathcal{H}$ is complete, the integration in the norm

$$||\phi||^2 = (\phi|\phi) = \int_{0\ \text{Lebesgue}}^{\infty} dE\ |\phi(E)|^2 \tag{9b}$$

[4]Instead of $t = 0$ one could choose any *finite* $t = t_0$ and $\phi(t) = \mathrm{e}^{-iHt/\hbar}\phi(t_0)$.

and in the scalar product (9a) need to be defined in terms of *Lebesgue* integrals, not Riemann integrals. Since most physicists do not work with Lebesgue integrals; the complete Hilbert space is hardly ever used by physicists.

2.2. Dirac formalism and the Schwartz space boundary conditions

The Hilbert space $\mathcal{H}$ is not the most suitable space to use for the theory of quantum physics for the following reasons. Physicists use linear scalar product spaces in which the scalar product is defined by Riemann integrals $(\psi,\,\phi) = \int_{0\ \mathrm{Riemann}}^{\infty} dE \langle\psi\,|\,E\rangle\langle E\,|\phi\rangle$. These spaces are not complete.[5] A scalar product space (or linear topological space) is complete if every Cauchy sequence has a limiting element in the space. This is not the case if norm and scalar product are defined by Riemann integrals and convergence is defined with respect to the norm; i.e.,

$$\begin{aligned} &\phi_n \to \phi \qquad \text{iff} \quad \|\phi_n - \phi\| \to 0 \quad \text{for } n \to \infty \\ \text{where} \quad & \\ &\|\phi\|^2 = (\phi, \phi) = \int_{\mathrm{Riemann}} dE\, \overline{\phi(E)}\, \phi(E)\,. \end{aligned} \tag{10}$$

In order to keep using Riemann integrals for the scalar product $(\cdot\,,\cdot)$ one cannot define the meaning of convergence by one norm or scalar product, it has to be defined in a different way.

Following Dirac (1925), physicists use Dirac kets which are not defined in Hilbert space. Dirac kets $|E\rangle$ have shown the way towards spaces which are complete spaces and in which the scalar product can be defined by Riemann integrals. Dirac [2] used the kets $|E\rangle$ to write $\phi(E)$ as $\langle E\,|\phi\rangle$ and $\overline{\psi(E)}$ as $\langle\psi\,|E\rangle = \overline{\langle E\,|\psi\rangle}$ and treated the integral as Riemann integral.

It took about 20 years to give a mathematical meaning to the Dirac kets $|E\rangle$. By 1950 L. Schwartz had created the theory of distributions [3] and Dirac kets $|E\rangle$ were defined as continuous antilinear functionals on the Schwartz space $F_E(\phi) = \langle E\,|\phi\rangle$. In the Schwartz space, usually denoted by Φ, the convergence of vectors is defined not by one scalar product as in (10), but by a *countable number* of scalar products [4]. One can justify most of Dirac's formalism of kets and bras [2], using the mathematics of locally convex linear topological spaces [3, 5, 4, 6] and their continuous functionals.

According to the Dirac formalism, an observable A (e.g., $A = H$) has a system of eigenvectors

$$H \mid E_n) = E_n \mid E_n) \quad \text{for discrete eigenvalue } E_n \tag{11a}$$

$$H\,|E\rangle = E\,|E\rangle \qquad \text{for continuous eigenvalue } E\,, \tag{11b}$$

and every state vector ϕ, fulfilling (4a) or vector ψ fulfilling (5b) can be expanded with respect to the energy kets (11) and/or with respect to eigenkets of other observables A. The Dirac basis vector expansion of state vector ϕ is

$$\phi = \sum_{n=\mathrm{integer}} \mid E_n)(E_n\,|\phi\rangle + \int dE\,|E\rangle\langle E\,|\phi\rangle\,, \tag{12a}$$

[5]With respect to the norm-convergence of the Hilbert space defined with (9a).

or, if there are *no* discrete eigenvectors $| E_n)$ representing bound states – the case discussed in these notes – the basis vector expansion is

$$\phi = \int_0^\infty dE \, |E\rangle\langle E \, |\phi\rangle \, . \tag{12b}$$

The eigenvectors $| E_n)$ for discrete eigenvalues E_n fulfill the orthonormality condition

$$(| E_n), | E_m)) \equiv (E_n \mid E_m) = \delta_{nm}. \tag{13}$$

The eigenvectors $|E\rangle$ for the continuous eigenvalue expansion (12b) were postulated to fulfill the new orthogonality condition called "Dirac orthogonality condition" [2]

$$\langle E \, |E'\rangle = \delta(E - E') \, , \tag{14}$$

where $\delta(E - E')$ is defined as the mathematical entity which fulfills the identity

$$\int_{-\infty}^{+\infty} dE' \langle E \, |E'\rangle\langle E' \, |\phi\rangle = \langle E \, |\phi\rangle \tag{15a}$$

$$\int_{-\infty}^{+\infty} dE' \delta(E - E')\phi(E') = \phi(E) \tag{15b}$$

for the set $\{\phi(E)\}$ of "well-behaved" function $\phi(E) = \langle E \, |\phi\rangle$.

Well-behaved means that $\phi(E)$ is infinitely differentiable and rapidly decreasing for increasing $|E|$. This set of functions is the Schwartz function space, $\{\phi(E)\} \equiv S$, of rapidly decreasing and infinitely differentiable functions [3, 5].

This Schwartz function space S is a dense subspace of the space L^2 of *Lebesgue* square integrable function: $S \subset L^2$.[6] This means that all functions $\phi(E) \in S$ are members of the subset of some *classes* of L^2-functions, i.e., $\phi(E) \in L^2$.[7] But in addition to these classes with $\phi(E) \in S$, there are sets of functions $\{h(E)\} \in L^2$ which contain no Schwartz space function. Thus $S \subset L^2$. Since according to the Fréchet-Riesz theorem: $L^2 = (L^2)^\times$, (where $(L^2)^\times$ denotes the space of antilinear Hilbert space- continuous functionals on L^2). It follows that one has the triplet of function spaces [5, 4, 6]:

$$\{\phi(E)\} = S \subset L^2 = (L^2)^\times \subset S^\times \, . \tag{16}$$

The Dirac δ-"function" $\delta(E - E') \in S^\times$ is not a function, like a well-behaved $\phi(E) \in S$, but a "distribution" defined by its property (15) for all $\phi(E) \in S$ (Schwartz space).

Here $(L^2)^\times$ and $S^\times$ denote the linear spaces of *continuous* antilinear functionals on L^2 and on S, respectively. The triplet (16) is the Rigged Hilbert Space (RHS) of Schwartz space functions. It gives a mathematical meaning to the Dirac kets $|E\rangle \in \Phi^\times$, as continuous, antilinear functionals on Φ.

[6] This means starting with S (smooth, rapidly decreasing functions) and adjoining to S all limit points of Cauchy sequences with respect to the Hilbert space convergence, one obtains L^2.

[7] The element of L^2 is not a function but a class of Lebesgue square integrable functions. Some of these classes contain a continuous rapidly decreasing function $\phi(E)$ which is an element of S.

The abstract Schwartz space Φ is the set of vectors $\{\phi\}$ of (12b) for which the $\langle E\,|\phi\rangle$ fulfills $\langle E\,|\phi\rangle \in S$; it is according to (16) the dense subspace of the abstract Hilbert space $\mathcal{H}$: $\{\phi\} \equiv \Phi \subset \mathcal{H}$. The space Φ has a stronger definition of convergence (also called stronger topology τ_Φ) than the Hilbert space convergence[8] $\tau_{\mathcal{H}}$; this means [4]:

$$\text{from } \phi_\nu \xrightarrow{\tau_\Phi} \phi \text{ with respect to } \tau_\Phi \text{ follows } \phi_\nu \xrightarrow{\tau_{\mathcal{H}}} \phi \text{ with respect to } \tau_{\mathcal{H}}, \text{ but not vice versa.} \tag{17}$$

Therefore $\Phi \subset \mathcal{H}$ and consequently for the continuous functionals $\mathcal{H}^\times \subset \Phi^\times$.

Therefore, in correspondence to the triplet of the Schwartz space functions (16), one obtains the triplet of the abstract Schwartz space, called the Rigged Hilbert Space (RHS) or Gelfand Triplet:[9]

$$\{\phi\} = \Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times\,. \tag{18}$$

The Schwartz space Φ is a nuclear space; this means for the Schwartz space, the Dirac basis vector expansion (12b) hold as the nuclear spectral theorem [3, 5, 4]. This theorem states that every vector $\phi \in \Phi$ can be expanded with respect to a complete set of generalized eigenvectors $|E\rangle \in \Phi^\times$ in a unique way

$$\phi = \int dE\; |E\rangle\langle E\,|\phi\rangle, \tag{19a}$$

$$\text{and } \phi = 0 \text{ if and only if } \phi(E) = \langle E\,|\phi\rangle = 0 \text{ for all } E\,. \tag{19b}$$

This justifies Dirac's expansion (12):
There exists a complete set $|\,E\rangle$ of eigenkets $|\,E\rangle \in \Phi^\times$ which are generalized eigenvectors of the Hamiltonian H with continuous eigenvalues $E \in \mathbb{R}$; i.e.,

$$\begin{matrix} H^\times\,|E\rangle = E\,|E\rangle \\ |E\rangle \in \Phi^\times \end{matrix} \qquad \text{precisely} \qquad \begin{matrix} \langle H\psi\,|E\rangle \equiv \langle\psi|\; H^\times\,|E\rangle = E\langle\psi\,|E\rangle \\ \text{for all } \psi \in \Phi\,, \end{matrix} \tag{20}$$

such that every $\phi \in \Phi$ can be expanded with respect to the $|E\rangle$ as in (19).

The operator H is "essentially self-adjoint" and $H^\times \supset H^\dagger = \overline{H} \supset H$ is the unique extension of $H^\dagger$ to the conjugate space $\Phi^\times$ (the space of all antilinear continuous functionals of the Schwartz space Φ). Therewith Dirac eigenket expansion (12b) has been given a mathematical meaning (19).

The abstract Schwartz space Φ is a linear topological space with the convergence defined by a countable number of norms $||\phi||_p$. $p = 1\,,2\,,\ldots$. E.g., for the oscillator, these norms are given by:

$$\begin{aligned} &||\phi||_p = (\phi|(N+1)^p|\phi) \quad \text{where } N = \tfrac{1}{\hbar\,\omega}(H - 1/2) \\ &\text{and} \\ &\phi_\nu \to \phi \text{ for } \nu \to \infty \text{ means } ||\phi_\nu - \phi||_p \to 0 \text{ for } \nu \to 0 \textit{ for all } p\,. \end{aligned} \tag{21}$$

[8]The convergence in Φ is defined by a countable number of norms, e.g., (21) below.
[9]The triplet (16) of function spaces is a "realization" of the RHS (15b) by function spaces in the same way as the coordinates x^i $(i = 1, 2, 3)$ are "realizations" of the vector $\overrightarrow{x} = \sum_{i=1}^{3} \overrightarrow{e_i}\, x^i$ in a 3-dimension space.

These countable norms are chosen such that the algebra of observables called $\mathcal{A}$ (in the case (21) for the oscillator, the algebra is generated by the momentum P, position Q, and the energy operator H) is represented by *continuous* operators in all of the space Φ.[4]

But the algebra of observables of a quantum physical system can usually not be represented by an algebra of continuous operators in $\mathcal{H}$ (e.g., momentum P and position Q of the oscillator are not continuous operators in Hilbert space $\mathcal{H}$).

Using the Schwartz space (18) and Dirac's bra-ket formalism, the set of vector-states $\{\phi\}$ fulfilling (4a) and the set of vector-observables $\{\psi\}$ fulfilling (5b) are both described by the same Schwartz space Φ which is a dense subspace of the Hilbert space $\mathcal{H}$ (Φ differs from $\mathcal{H}$ by limit points of Hilbert space Cauchy sequences).

One can now ask for all solutions $\phi(t)$ of the Schrödinger equation (4a) under the Schwartz space boundary condition. Similarly, one can ask for all solutions $\psi(t)$ of the Heisenberg equation (5b) under the Schwartz space boundary condition:

$$\text{Set of state vectors } \{\phi\} = \Phi = \text{Schwartz space} \subset \mathcal{H} \subset \Phi^{\times} \tag{22a}$$

$$\text{Set of observable vectors } \{\psi\} = \Phi = \text{Schwartz space} \subset \mathcal{H} \subset \Phi^{\times} \tag{22b}$$

Requiring this Schwartz space boundary conditions (22) for the dynamical equation (4a) and (5b), means only the vectors $\phi \in \Phi$ (not all vector of $\mathcal{H}$) represent physical states prepared, (e.g., by a preparation device or preparation apparatus in an experimental setup) and the same set of vectors $\psi \in \Phi$ represent also the observables detected by the registration apparatus, e.g., a detector.

Using equations (22) as an axiom for the solutions for the Schrödinger equation (4a) and for the Heisenberg equation (5b) one obtains by a mathematical theorem (Proposition II page 82 of [6]) (like the Stone-von Neumann result (7),) that the time evolution is given by a group, (the unitary group in (7) restricted to Φ). This means for the solution of the Schrödinger equation (4a) under the boundary conditions (22a) one obtains

$$\phi(t) = U_{\Phi}^{\dagger}(t)\, \phi = \mathrm{e}^{-iHt/\hbar}\phi \quad -\infty < t < +\infty \tag{23a}$$

And for the solutions for the Heisenberg equation (5b) under the boundary conditions (22b) one obtains

$$\psi(t) = U_{\Phi}(t)\, \psi = \mathrm{e}^{iHt/\hbar}\psi, \quad \text{with } -\infty < t < +\infty \tag{23b}$$

In (23), $U_{\Phi}^{\dagger}(t)$ and $U_{\Phi}(t)$ are the restriction of the unitary operator $U^{\dagger}(t)$ in (7a) and of $U(t)$ in (7b) to the dense Schwartz-subspace Φ of the Hilbert space $\mathcal{H}$: $U_{\Phi}^{\dagger}(t) = U_{\mathcal{H}}^{\dagger}(t)|_{\Phi}$ [6].

For the time evolution of the Dirac kets $|E\rangle$ (Schwartz space functional) one has then

$$|E\,;t\rangle = \mathrm{e}^{-iH^{\times}t/\hbar}\,|E\rangle = \mathrm{e}^{-iEt/\hbar}\,|E\rangle, \quad \text{with } -\infty < t < +\infty\,. \tag{24}$$

The Born probabilities (1), $\mathcal{P}_W(\Lambda(t))$, to measure an observable $\Lambda(t)$ in a state W under the Schwartz space axiom are thus again predicted for all t: $-\infty < t < +\infty$:

$$\mathcal{P}_W(\Lambda(t)) = \mathrm{Tr}(W\,\Lambda(t)) = \mathrm{Tr}(W(t)\,\Lambda) \quad \text{for all } -\infty < t < +\infty\,. \tag{25}$$

For the case that W is a pure state $W = |\phi\rangle\langle\phi\,|$ and the observable is given by $\Lambda = |\psi\rangle\langle\psi\,|$, this probability is written as

$$\begin{aligned}\mathcal{P}_\phi(|\psi\rangle\langle\psi\,|) &= \mathrm{Tr}(|\psi\rangle\langle\psi\,|\phi(t)\rangle\langle\phi(t)\,|) \\ &= |\langle\psi\,|\phi(t)\rangle|^2 = |\langle\psi(t)\,|\phi\rangle|^2 \quad \text{for all } \; -\infty < t < +\infty\,.\end{aligned} \tag{26}$$

This means that the theory based on the Hilbert space boundary condition (6) *as well as* the theory based on the Schwartz space boundary condition (22) *predict* a probability $|\langle\psi\,|\phi(t)\rangle|^2$ to detect the observable $\Lambda = |\psi\rangle\langle\psi\,|$ in the state $\phi(t)$, for arbitrary negative times, i.e., even for time *before* the state $\phi(t)$ had been prepared at the time $t = t_0 = 0$.

It is this kind of theorems, particularly the Stone-von Neumann theorem for the Hilbert space, which made us think that in quantum physics of scattering and decay, the time needs to extend from $-\infty < t < +\infty$.

2.3. A causality condition for quantum mechanics

In *contrast* to the mathematical prediction (26) for the Hilbert space boundary condition (6) as well as for the Schwartz space boundary condition (22), in the laboratory, the situation is quite different, because of the causality principle. This empirical principle states:

> A state ϕ needs to be prepared first at a time t_0 *before* an observable $|\psi(t)\rangle\langle\psi(t)\,|$ can be measured in that state ϕ with the probability $\mathcal{P}_\phi(|\psi(t)\rangle\langle\psi(t)\,|)$. (27)

The principle (27) means: *only* for times $t > t_0$, where t_0 is the time at which the state ϕ is prepared, can one *detect* the observable $|\psi(t)\rangle\langle\psi(t)\,|$ or any observables $A(t)$ in the state ϕ, but not at any arbitrary time $t < t_0$ in the distant past. Therefore, the time symmetric group evolution (7) as well as (23) – predicted by mathematics from the boundary condition (6) and also from the boundary condition (22) – is in contradiction with the causality principle (27). Causality (27) means that an observable cannot be detected in a state *before* this state exits, i.e., before it has been prepared (by a preparation apparatus or, may be, by a big bang): Born probabilities $\mathcal{P}_W(\Lambda(t))$ to measure observables $\Lambda(t)$ in states W make sense *experimentally* only for $t \geq t_0 =$ preparation time of the state W.

Therefore, a new boundary condition is needed in place of (22), or (6), which predicts the Born probabilities

$$\begin{aligned}\mathcal{P}_\phi(|\psi\rangle\langle\psi\,|) &= \mathrm{Tr}(|\psi\rangle\langle\psi\,|\phi(t)\rangle\langle\phi(t)\,|) \\ &= |\langle\psi\,|\phi(t)\rangle|^2 = |\langle\psi(t)\,|\phi\rangle|^2 \quad \textit{only} \text{ for } t > t_0\,.\end{aligned} \tag{28}$$

Here t_0 is a *finite time*, namely the time at which the state $\phi(t)$ had been prepared and after which an observable $|\psi\rangle\langle\psi\,|$ can be measured in this state.

Summarizing: In order to have a theory that agrees with causality as formulated by (27), one needs to find boundary conditions for the Schrödinger or for Heisenberg equation which predict the solutions of the dynamical equations (4a) or (5b) *only for* $t \geq t_0$, where t_0 is finite. Thus, one needs to solve the dynamical differential equation[10] under new boundary conditions which are based on the existence of this finite time t_0. The results will be *time asymmetric solutions* of the dynamical equations, which distinguish the finite time t_0.

The mathematics of this problem will be presented in the following section; it is based on the Hardy space boundary condition for the dynamical equation and constitutes no problem. The interpretation and the observation of this finite t_0 is an other matter and may require some introductory remarks:

As is usually the situation in quantum physics, where one does not deal with one quantum system but with an ensemble, this beginning of time t_0 is realized as an ensemble of times. Experiments are made on an ensemble of micro-system, and an ensemble of micro-systems is usually prepared at an ensemble of times $t_0^{(i)}$, on the clock at the laboratory walls (or even in different laboratories at different times). This ensemble of times $t_0^{(i)}$ is the preparation time t_0 of the state described by the vector $\phi(t) = e^{-i(t-t_0)H/\hbar}\ \phi(t_0)$; t_0 represents the ensemble of quantum systems in a pure state ϕ prepared at an ensemble of time $t_0^{(i)}$. Therefore, the time t_0 is also likely to be detected as an ensemble of times, and one should look for experiments where one could observe these times $\{t_0^{(i)}\}$. Furthermore, ensembles of a micro-systems of the same quantum system are often prepared at different times and often in different labs at different places. Still the $\{t_0^{(i)}\}$ are the beginnings of time for the ensemble of identical micro-system.

In conventional scattering theory, e.g., [7], one distinguishes between an in-state and an out-"state" vector. The in-state is prepared as an accelerator beam but the out-state is detected or registered by a registration apparatus. This means the so-called "out-state" is really a detected observable $\psi(t)$ and therefore should obey the Heisenberg equation (5b) and not the Schrödinger equation (5a) (as would be the standard interpretation for an in-state in conventional scattering theory). The accelerator *prepares* a state $\phi(t)$ while the registration apparatus (detector) registers the observable $\psi(t)$ by counts of a detector, thus $A(t) = |\psi(t)\rangle\langle\psi(t)|$ obeys the Heisenberg equation (5a) and $\psi(t)$ obeys the equation (5b).

From the above re-interpretation of the time evolution for experiments on an ensemble of quantum particles, we get the idea to use two different mathematical spaces: one for the space of prepared in-*states* ϕ or W, the other mathematical space for the space of detected out-*observables* ψ or A. This suggests that the mathematical theory for the scattering process needs to use two different mathematical spaces instead of the one Schwartz space Φ of (22), or the Hilbert space $\mathcal{H}$ of (6). In any case, the mathematical Hilbert space $\mathcal{H}$ is never used by physicists, except to justify the unitary group evolution (7a) or/and (7b) by the Stone-von Neumann theorem [1].

[10] In the Schrödinger picture or in the Heisenberg picture.

3. In-states, out-observables, and the Lippmann-Schwinger kets suggest the Hardy space axiom

In standard scattering theory [7] one speaks of in-states ϕ^{in} "controlled" in the remote past and of out-states ψ^{out} "controlled" in the distance future. *Both*, the in-state ϕ^{in}, as well as the out-states ψ^{out} are thought to obey the Schrödinger equation (4a) (under the Hilbert space boundary condition (6a)) for states. However, the controlled, so-called out-state vectors ψ are controlled by a registration apparatus (e.g., a detector). This means that the "controlled out vectors" *represent really observables registered by the detector*. Therefore, the so-called "out-state" is really an observable which should be governed by the Heisenberg equation (5b) with the solutions given by (7b) or (23b), not governed by the Schrödinger equation (7a).

Under the standard boundary condition (6b) and (22b), the solutions of the Heisenberg equation for the observables are predicted for all time t: $-\infty < t < +\infty$. This is in conflict with the causality principle (28): According to (28) the observable $|\psi\rangle\langle\psi|$ in the state ϕ can be predicted only for times $t > t_0$ where t_0 is the time at which ϕ has been prepared. To avoid violation of the causality principle, we need to find boundary conditions for the solutions of the dynamical equations (4) and (5), which will be different from the Hilbert space boundary condition (6) and also different from the Schwartz space boundary condition (22).

These new boundary conditions need to use different representation spaces than the Hilbert space $\mathcal{H}$ or the Schwartz space Φ. These new spaces we call (in anticipation of our conclusion):

Φ_- for the solutions of the Schrödinger equation of the states $\{\phi^+\}$.

Φ_+ for the solutions of the Heisenberg equation of the observables $\{\psi^-\}$.

This means, one needs to modify the Hilbert space axiom (6) of von Neumann. Similarly, the Schwartz space axiom of the Dirac formalism, which is summarized by the mathematical statement (22), has to be modified, if (26) is to be avoided and if the causality principle (28) is to be obeyed. Thus we replace the axiom (22) (or (6)) for the dynamical equations by a new axiom that *distinguishes mathematically* between:

The prepared states which are represented by the set of prepared in-state vectors $\{\phi^+\}$, obeying the Schrödinger equation (4a), and the detected observables which are represented by the set of registered out-observables $\{\psi^-\}$, obeying the Heisenberg equations (5b).

The $^+$ and $^-$ labels have been chosen to refer to the in-state vector $\{\phi^+\}$ and to the out-vectors $\{\psi^-\}$. This is the standard notation of scattering theory for the in-vectors ϕ^+ referring to the prepared states, and for the out-vector ψ^- or the operators $|\psi^-\rangle\langle\psi^-|$ referring to the detected observables[11].

---

[11]But the out-vectors (or so-called out-"states") can be many things and what one detects as the Born probability $|(\psi^-, \phi^+)|^2$ depends upon the choice of the particular registration apparatus (detector) which is built such that a particular property $|\psi^-\rangle\langle\psi^-|$ (or by a more general observable represented by $A^- = \sum_i \lambda_i |\psi_i^-\rangle\langle\psi_i^-|$ or by $\int d\lambda\, |\psi_\lambda^-\rangle\langle\psi_\lambda^-|$) is detected.

The Dirac basis vector expansion[12] of in-state vectors $\phi^+ \in \Phi_-$ is given by

$$\phi^+ = \sum_b \int_0^\infty dE|E, b^+\rangle\langle^+E, b|\phi^+\rangle = \int_0^\infty dE|E^+\rangle\langle^+E|\phi^+\rangle \tag{29a}$$

And for the out-observable vector ψ^- the Dirac basis vector expansion is given by

$$\psi^- = \sum_b \int_0^\infty dE|E, b^-\rangle\langle^-E, b|\psi^-\rangle = \int_0^\infty dE|E^-\rangle\langle^-E|\psi^-\rangle \tag{29b}$$

Here the energy eigenkets $|E^+\rangle \in \Phi_-^\times$ are continuous antilinear functionals on the space Φ_- of prepared states. They fulfill

$$\langle H\phi^+|E^+\rangle = \langle\phi^+|\, H^\times\, |E^+\rangle = E\langle\phi^+|E^+\rangle \quad \text{for all} \quad \phi^+ \in \Phi_-\,, \tag{30a}$$

Similarly, the $|E^-\rangle \in \Phi_+^\times$ of (29b) are continuous antilinear functions on Φ_+:

$$\langle H\psi^-|E^-\rangle = \langle\psi^-|\, H^\times\, |E^-\rangle = E\langle\psi^-|E^-\rangle \quad \text{for all} \quad \psi^- \in \Phi_+\,. \tag{30b}$$

Though the mathematical spaces Φ_-, Φ_+ had not been defined previously, the kets $|E^+\rangle$ and $|E^-\rangle$ have been used extensively for a long time in scattering theory. They are the Lippmann-Schwinger kets [8] of (31) below.

Since the space of in-states Φ_- and the space of out-observables Φ_+ are different subspaces of $\mathcal{H}$, the nuclear spectral theorem for the basis vector expansion (29a) of in-states $\phi^+ \in \Phi_-$ and (29b) of out-observables $\psi^- \in \Phi_+$ require that each space has its own basis. In (29a), the basis kets for Φ_- have been denoted by $|E^+\rangle \in \Phi_-^\times$ and the basis kets in (29b) for Φ_+ have been denoted by $|E^-\rangle \in \Phi_+^\times$. The question now is: What is the mathematical space Φ_- which represent the in-states $\{\phi^+\}$ and what is the mathematical space Φ_+ which represent the out-observables $\{\psi^-\}$? They will turn out to be the pair of Hardy spaces [6, 9, 10, 11, 12, 13, 14, 15, 16].

Before the Hardy spaces were used in quantum physics, kets like the $|E^\pm\rangle$ had been introduced in the phenomenological scattering theory as the in- and out-plane wave kets $|E^\pm\rangle$ which fulfill the Lippmann-Schwinger equation [7, 9, 8]:

$$|E, b^\pm\rangle = |E \pm i\epsilon, b^\pm\rangle = |E, b\rangle + \frac{1}{E - H \pm i\epsilon}V|E, b\rangle = \Omega^\pm|E, b\rangle, \quad \epsilon \to +0\,. \tag{31}$$

The kets $|E, b^\pm\rangle$ are eigen-kets of the "exact" Hamiltonian $H = K + V$,

$$H\,|E, b^\pm\rangle = E_b\,|E, b^\pm\rangle = E_b\,|E, j, j_3, \eta^\pm\rangle\,, \tag{32}$$

and the kets $|E,\, b\rangle$ in (31) are the eigen-kets of the interaction free Hamiltonian K: $K\,|E,\, b\rangle = E\,|E,\, b\rangle$. The label b represents additional quantum numbers such as the angular momentum number j, its third component j_3, and other quantum number, e.g., channel quantum numbers η or particle species labels, *etc.* ... The operator $V = H - K$ is the interaction Hamiltonian or perturbation Hamiltonian, and $\Omega^\pm$ are the Möller operators [7, 9].

[12] Though the properties of the spaces are not yet known, the assumption that $\Phi_\pm$ will be nuclear spaces is reasonable, so that the nuclear spectral theorem (29) will hold.

The Lippmann-Schwinger kets (31)(32) need to be given a mathematical definition before one can use them to calculate *mathematical* predictions.

The $+i\epsilon$ in the ket $|E^+\rangle = |E+i\epsilon\rangle$ of the Lippmann-Schwinger equation (31) suggested that the energy wave function of the in-states ϕ^+ of (29a),

$$\phi^+(E) \equiv \langle^+E|\phi^+\rangle = \langle^+E,b|\phi^+\rangle = \overline{\langle\phi^+|E,b^+\rangle} \tag{33a}$$

are the boundary value of an analytic function in the lower complex energy semi-plane $\mathbb{C}_-$ on the second sheet of the $\mathcal{S}$-matrix, cf. Figure 1.

Similarly, the $-i\epsilon$ sign of the Lippmann-Schwinger ket $|E^-\rangle = |E-i\epsilon\rangle$ indicates that the energy wave function of the observable $|\psi^-\rangle\langle\psi^-|$ in (29b),

$$\psi^-(E) \equiv \langle^-E|\psi^-\rangle = \langle^-E,b|\psi^-\rangle = \overline{\langle\psi^-|E,b^-\rangle} \tag{33b}$$

are the boundary value of an analytic function on the upper complex energy semi-plane $\mathbb{C}_+$ for complex energy $z = \overline{E-i\epsilon} = E+i\epsilon$, above the real axis of the second sheet of the $\mathcal{S}$-matrix. Consequently, its complex conjugate $\overline{\psi^-(E)} = \langle\psi^-|E,b^-\rangle$ is analytic on the lower complex energy plane second sheet of $\mathcal{S}$-matrix.

This means that the energy density $\langle\psi^-|E^-\rangle\mathcal{S}_j(E)\langle^+E|\phi^+\rangle$ in the $\mathcal{S}$-matrix element (ψ^-,ϕ^+):

$$(\psi^-,\phi^+) = \int_{E_0(=0)}^{\infty} dE\langle\psi^-|E^-\rangle\mathcal{S}_j(E)\langle^+E|\phi^+\rangle \tag{34}$$

can be analytically continued into the lower complex energy plane second sheet of the $\mathcal{S}$-matrix as shown in Figure 1. This is the sheet at which the resonance pole of the $\mathcal{S}$-matrix element $S_j(E)$ is located at z_R.

This means that the space $\Phi_\mp$ of the states $\{\phi^\pm\}$ will be mathematically defined as Hardy space of the lower and upper complex semi-plane, respectively. The $|E^\pm\rangle$ will now be defined as Hardy space functionals[13]: $|E^\pm\rangle \in \Phi_\mp^\times$. The new axiom replacing the Hilbert space axiom (6), or replacing the Schwartz space axiom (22) for the Dirac formulation, is now introduced as the new *Hardy space axiom* of quantum mechanics:

> The set of prepared (in-)states obeying Schrödinger equation $\{\phi^+\}$ is mathematically represented by Φ_-, the Hardy space of the lower complex energy plane of the second sheet of the $\mathcal{S}$-matrix:
>
> $$\{\phi^+\} \doteq \Phi_- \,. \tag{35a}$$
>
> The set of detected or registered observables obeying Heisenberg equation $\{\psi^-\}$ is mathematically represented by Φ_+, the Hardy space of the upper complex energy plane of the second sheet of the $\mathcal{S}$-matrix:
>
> $$\{\psi^-\} \doteq \Phi_+ \,. \tag{35b}$$

[13]The odd notation $|E^\pm\rangle \in \Phi_\mp^\times$ comes from the miss-match of the notation which the physicists use for the phenomenological Lippmann-Schwinger kets of (31) and the mathematical convention for the Hardy spaces [12, 13]. That the phenomenologically introduced Lippmann-Schwinger kets $|E^\pm,b\rangle$ of scattering theory [8] turned out to be anti-linear continuous functionals on Hardy space [9, 10, 13], is an example of what Wigner [17] called "The Unreasonable Effectiveness of Mathematics in the Natural Sciences".

This Hardy space axiom means that the energy wave functions $\phi^+(E) = \langle {}^+E\,|\phi^+\rangle$ and $\psi^-(E) = \langle {}^-E\,|\psi^-\rangle$ in the Dirac basis vector expansion (29) are not just functions of the Schwartz space, but that $\phi^+(E)$ can also be analytically continued into the lower complex energy plane second sheet of the $\mathcal{S}$-matrix and $\psi^-(E)$ can be analytically continued into the upper complex plane. Therefore $\langle\psi^-\,|E^-\rangle\,\langle{}^+E\,|\phi^+\rangle = \overline{\psi^-(E)}\,\phi^+(E)$, which appears in the $\mathcal{S}$-matrix element (34), can be analytically continued into the second sheet, where the poles of the $\mathcal{S}$-matrix, which represent resonances (with angular momentum j), are located; cf. Figure 1, where the special case of one resonance pole is considered.

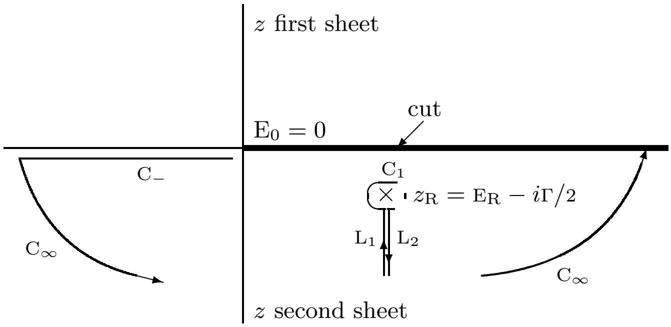

FIGURE 1. Complex energy plane in which one first-order pole z_R is located on the second sheet of the S_j-matrix.

In terms of the energy wave functions of the Dirac basis vector expansion (29), the Hardy space axiom (35) is also stated as:

$$\phi^+(E) = \langle {}^+E|\phi^+\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+})^\times \tag{36a}$$

$$\psi^-(E) = \langle {}^-E|\psi^-\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+})^\times\,. \tag{36b}$$

Here $(\mathcal{H}^2_\pm \cap S)_{\mathbb{R}_+}$ denotes the space of Hardy classes $\mathcal{H}^2_\pm$ intersected with the Schwartz space function S and then restricted to the positive semi-axis $\mathbb{R}_+$, where the cut of the $\mathcal{S}$-matrix is located, cf., Figure 1.

The Hardy space axiom in the energy representation (36) thus says vaguely that the energy wave functions are very well-behaved functions: Schwartz functions that can also be analytically continued into the complex energy plane second sheet of the $\mathcal{S}$-matrix element $\mathcal{S}_j(E)$ of angular momentum j, where the resonance poles of the $\mathcal{S}$-matrix are located; cf., Figure 1.

We consider the case that there is one resonance pole at z_R, as shown in Figure 1. One can deform the contour of integration in (34) from the cut along the positive real energy axis $0 \le E < \infty$ to an integral around the resonance pole z_R and an integral along the contour C_- (the integrals along L_1 and L_2 cancel each other and the integral along C_∞ vanishes as a consequence of the Hardy space properties (36)).

The new Hardy space axioms (35) conjectured from the phenomenological Lippmann-Schwinger equation (31), suggests that the energy wave functions of the prepared in-state $\phi^+(E)$ and the energy wave functions of the detected out-observable $\psi^-(E)$ are not only smooth, rapidly decreasing and infinitely differentiable functions on the real axis, as they would be under the Schwartz space axiom (22). But $\phi^+(E)$ and $\psi^-(E)$ are also analytic in the lower complex energy semi-plane on the *second* sheet of the $\mathcal{S}$-matrix, where a resonance pole of the $\mathcal{S}$-matrix is located, in Figure 1. These functions are by axiom (36) *postulated* to be *smooth* Hardy functions $\mathcal{H}^2_\mp \cap S$ of the lower and upper complex plane second sheet of the $\mathcal{S}$-matrix, restricted to the positive real axis. These energy wave functions of the in-state $\phi^+(E) = \langle^+E\,|\phi^+\rangle$ and of the out-observable $\psi^-(E) = \langle^-E\,|\psi^-\rangle$ are elements of the spaces which are an intersection of two space: the Schwartz space S and the upper and lower Hardy class space $\mathcal{H}^2_\mp$, respectively

$$\langle^+E\,|\phi^+\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+} \;\text{ or }\; \overline{\langle^+E\,|\phi^+\rangle} = \langle\phi^+\,|E^+\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+} \tag{37a}$$

$$\langle^-E\,|\psi^-\rangle \in (\mathcal{H}^2_+ \cap S)\,|_{\mathbb{R}_+} \;\text{ or }\; \overline{\langle^-E\,|\psi^-\rangle} = \langle\psi^-\,|E^-\rangle \in (\mathcal{H}^2_- \cap S)\,|_{\mathbb{R}_+}\,. \tag{37b}$$

This means that the energy wave functions are smooth Hardy class functions $\mathcal{H}_\mp$ which are also in Schwartz spaces, i.e., they are smooth rapidly decreasing functions on the positive real axis $\mathbb{R}_+$, which can be analytically continued into the upper or lower complex semi planes and vanish rapidly going towards the infinite semi-circles C_∞. The conditions (37), which in the vector notation are the conditions (35), constitute an axiom, (which we call the Hardy space axiom). Like the Hilbert space axiom (6), these kinds of axioms can only be justified by its success with experimental data.

The smooth Hardy space wave functions of (37) posses many properties needed for the analytic $\mathcal{S}$-matrix and the phenomenological theory of resonances and decay. In this paper, we conjectured their property from the Lippmann-Schwinger equation (31), re-interpreting the out-plane wave $|\,E^-\rangle$ as the kets for an out-*observable* $|\,\psi^-\rangle$ which obeys the Heisenberg equation (5b), *not* the Schrödinger equation (4a) as usually assumed. With the Hardy space properties (37) of the energy wave functions, one can associate to the $\mathcal{S}$-matrix pole a resonance state vector [9]. All this is fine and fits well together, with the conventional ideas except for one shocking consequence: the time-asymmetry that will result if one solves the Schrödinger or the Heisenberg equation under Hardy space boundary condition (35) or (36).

4. Conclusion: Time asymmetry of quantum physics from the Hardy space axiom

Solutions of differential equations require *boundary conditions*, which specify the properties that these solutions will fulfill. In the same way as the unitary group evolution (7a) for the solutions of the dynamical equations (4a) and (5b) follow from the Hilbert space boundary condition by the Stone-von Neumann theorem[14], there is a similar theorem in the mathematical literature from which the solutions of (4a) and (5b) follow under the Hardy space boundary conditions (35). This theorem is the Paley-Wiener theorem [18], and from the Paley-Wiener theorem follows that the solutions of the Schrödinger equation (4a) under the new Hardy space boundary condition (35a) are given by the *semigroup* of operator $U_{-}(t)$:

$$\phi^{+}(t) = U^{\dagger}_{\Phi_-}(t)\,\phi^{+} = e^{-iHt/\hbar}\,\phi^{+}(0) \quad \text{with } 0 \leq t < \infty \quad \text{for } \phi^{+} \in \Phi_{-}\,. \tag{38a}$$

Similarly, by the Paley-Wiener theorem, the solution of the Heisenberg equation (5b) are given by the *semigroup* of operator $U_{+}(t)$[15]

$$\psi^{-}(t) = U_{\Phi_+}(t)\,\psi^{-} = e^{iHt/\hbar}\,\psi^{-}(0) \quad \text{with } 0 \leq t < \infty \quad \text{for } \psi^{-} \in \Phi_{+}\,. \tag{38b}$$

As a consequence of (38) follows:

The Born probabilities to detect the observable $\psi^{-}(t)$ in the state ϕ^{+} under Hardy space boundary conditions are given by

$$\begin{aligned} \mathcal{P}_{\phi^+}(\psi^{-}(t)) &= |\langle\psi^{-}(t)\,|\phi^{+}\rangle|^2 = |\langle\psi^{-}\,|\phi^{+}(t)\rangle|^2 = |\langle e^{iHt/\hbar}\psi^{-}\,|\phi^{+}\rangle|^2 \\ &= |\langle\psi^{-}\,|e^{-iHt/\hbar}\phi^{+}\rangle|^2 \quad \textit{for only } t \geq t_0 = 0 \end{aligned} \tag{39}$$

This means that from the Hardy space boundary condition (35a) follows the semigroup time evolutions (38a) for the solutions in the Schrödinger picture. Or similarly in the Heisenberg picture, from the Hardy space axiom (35b) follows the semigroup evolution (38b) for the observables. Therefore, the Born probabilities (39) are predicted under the Hardy space axiom only for $t \geq t_0$, i.e., only for a time t after the *finite* time t_0 at which the state has been prepared. This prediction is in agreement with the causality principle (27) and (28).

The time t_0 is chosen as the finite time $t_0 = 0$. It represents the time at which the state ϕ^{+} has been prepared, e.g., by an accelerator beam and target, and after which the observable ψ^{-} can be registered, e.g., by a detector with the counting rates $N(t)/N$ proportional to the probability (39).

Solutions of differential equations require boundary conditions, which specify general properties that these solutions are to fulfill. The traditional boundary conditions (6a) for the Schrödinger- and equation (6b) for the Heisenberg-equation require, that these solutions are the (complete) Hilbert space (with the scalar product defined by the Lebesgue integral, von Neumann's great contribution to quantum mechanics). If one would use these Hilbert space boundary conditions (6), the solutions would be given by the time symmetric, unitary group evolution (7), according to the famous theorem for the Hilbert space by Stone and von Neumann [1].

Using, instead of the Hilbert space, the Schwartz space boundary conditions (22) of the Dirac formulation, one also obtains a time symmetric group evolution

[14] And the group evolution for the Schwartz space boundary condition followed from another mathematical theorem (page 82 [6])).

[15] The operators $U_{\pm}(t)$ form semigroups since their inverse operators $U^{-1}_{\pm}(t)$ do not exist.

(23) by another mathematical theorem ([6, Prop. II p. 82]. But the Hilbert space and the Schwartz space are not the only possible boundary conditions for the dynamical differential equations (4a) and (5b) of quantum mechanics.

The Hardy spaces used in the Lax-Phillips scattering theory [19], were applied to scattering of classical (e.g., electromagnetic) waves. They have also been applied to quantum resonance and decay phenomena [14, 15, 16, 20, 21, 22].

In this paper we have discussed how different boundary conditions for the dynamical equations lead to different time evolution, the unitary group (23) and the semigroup evolution (38) for Hardy spaces, which is the only axiom compatible with causality.

The Hardy space axiom (35) or (33) provides the mathematical theory [22] that associates as a mathematical relation, the first-order poles of the $\mathcal{S}$-matrix at the complex energy $z_R = E_R - i\Gamma/2$ on the second sheet of the $\mathcal{S}$-matrix (Figure 1), a generalized eigenvector $|z_R^-\rangle = |E_R - i\Gamma/2^-\rangle \in \Phi_+^\times$ of the Hamiltonian H with the complex eigenvalue $z_R = E_R - i\Gamma/2$ and with a Breit-Wigner resonance distribution of width Γ:

$$|z_R^-\rangle\sqrt{2\pi\Gamma} = -\frac{1}{i}\left(\frac{\Gamma}{2\pi}\right)^{1/2}\int_{-\infty_{II}}^{+\infty} dE\ |E^-\rangle\frac{1}{E - z_R}\,. \tag{40}$$

From this one calculates the probability (39) for an observable $\psi^-(t)$ of (38b) in the "first-order-$\mathcal{S}$-matrix-pole-state" $|z_R^-\rangle$

$$\begin{aligned}\mathcal{P}_{z_R^-}(\psi^-(t)) &= |\langle\psi^-(t)\,|z_R^-\rangle|^2 = |\langle e^{iHt/\hbar}\,\psi^-\,|E_R - i\Gamma/2^-\rangle|^2\\ &= |\langle\psi^-\,|\ e^{-iH^\times t/\hbar}\ |E_R - i\Gamma/2^-\rangle|^2\\ &= |e^{-iE_R t/\hbar}\ e^{-(\Gamma/2)t/\hbar}\langle\psi^-\,|E_R - i\Gamma/2^-\rangle|^2\\ &= e^{-\Gamma t/\hbar}\ |\langle\psi^-\,|z_R^-\rangle|^2\ \ \text{for } t \geq 0 \text{ only for all } \psi^- \in \Phi_+\,.\end{aligned} \tag{41}$$

To a $\mathcal{S}$-matrix pole resonance of width Γ is associated a state vector $|\,z_R = E_R - i\Gamma/2^-\rangle$ with exponential time evolution of lifetime $\tau = \hbar/\Gamma$ [9]. The Hardy space boundary condition for the dynamical equations provides the mathematical theory that unifies resonance and decay phenomena of quantum physics.

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Factorization Method and the Position-dependent Mass Problem

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To Professor Bogdan Mielnik, for all his contributions in Physics

Abstract. The dynamics of position-dependent mass systems is considered from both, classical and quantum mechanical points of view, by means of the factorization method. Some examples are presented, with particular choices of the mass function, for the harmonic oscillator in order to illustrate our results. In the quantum regime, new isospectral position-dependent mass potentials are also constructed by the intertwining technique.

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1. Introduction

The problem of describing the motion of systems endowed with position-dependent mass (PDM) has attracted interest since they appear in many physical problems. These include, *e.g.*, the study of the electronic properties of semiconductors [1–3], quantum dots [4], the description of the dynamics of non linear oscillators [5, 6] as well as classical systems in curved spaces [7], just to mention few ones. The very concept of a PDM system is a fundamental problem which is far from being completely understood. Many contributions have been developed over the last years in different approaches [8–19]. In the quantum mechanical regime, it is well known that an ambiguity in ordering of the mass and the momentum operators appears and the goal is to choose the proper Hamiltonian. Some arguments have been given to this respect, e.g., the Galilean invariance [8] and the correspondence between classical and quantum PDM potentials [16]. In some other cases the ordering is fixed by the boundary conditions imposed on a particular system [19]. The generation of exactly solvable PDM problems has also been considered. The factorization method [20–22] has been explored in [10–17]. In this work we present the factorization method applied to the solution of the PDM problem in the classical as well as in the quantum mechanical frames. The paper is organized as follows. In Section 2 the classical case is considered and some examples are presented for the harmonic oscillator algebra. In Section 3 the quantum mechanical problem is discussed and some new PDM potentials isospectral to the harmonic oscillator are constructed. We end this contribution with some general remarks.

2. Classical position-dependent mass systems

Consider the classical position-dependent mass system described by the standard Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m(x)} + \mathcal{V}(x) \tag{1}$$

where x and p are the canonical variables of position and linear momentum. The mass $m(x) > 0$ and the potential $\mathcal{V}(x)$ are position-dependent functions setting the domain of definition $\mathcal{D}(\mathcal{H})$ of the Hamiltonian. The problem can be addressed from two points of view: in the first one $\mathcal{V}$ and m are known and the phase space motion is determined by reducing the PDM problem to an equivalent CM one; in the second case, it is assumed that there is an algebraic structure fixing the potential and the phase space trajectories in terms of $m(x)$ [23] (see also [16]). In this work, the second approach is considered: the explicit form of the potential as well as the dynamics are determined from the algebraic properties of the system, by the factorization method.

Suppose that the Hamiltonian $\mathcal{H}$ can be factorized in terms of two complex functions [23]

$$\mathcal{A}^{\pm} = \mp i f(x) \frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x)\phi(\mathcal{H}) \tag{2}$$

in the form

$$\mathcal{H} = \mathcal{A}^{-}\mathcal{A}^{+} + \epsilon = \mathcal{A}^{+}\mathcal{A}^{-} + \epsilon, \tag{3}$$

with ϵ the factorization constant, f, $\mathcal{W}$ functions of the position and ϕ a function of the energy of the system.

Suppose, additionally, that $\mathcal{A}^{\pm}$, $\mathcal{H}$ close the following algebra in terms of Poisson brackets

$$\{\mathcal{A}^{-}, \mathcal{A}^{+}\} = i\gamma\phi(\mathcal{H}), \quad \{\mathcal{A}^{\pm}, \mathcal{H}\} = \pm i\gamma\phi(\mathcal{H})\mathcal{A}^{\pm}, \tag{4}$$

where γ is a constant. Observe that two complex-conjugate, non autonomous integrals of motion can be constructed in the form

$$\mathcal{Q}^{\pm} = \mathcal{A}^{\pm} e^{\mp i\gamma\phi(\mathcal{H})t}, \tag{5}$$

whose values $q^{\pm}$ fulfill $q^{-}q^{+} = |q^{\pm}|^2 = \mathcal{E} - \epsilon$, $\mathcal{E}$ being the total energy of the system. Thus, making $q^{\pm} = \sqrt{\mathcal{E} - \epsilon}\, e^{\pm i\varphi_0}$, the phase space trajectories can be written in terms of two parameters $(\mathcal{E}, \varphi_0)$ as

$$x(t) = \mathcal{W}^{-1}\left(\frac{\sqrt{\mathcal{E}-\epsilon}}{\phi(\mathcal{H})}\cos\left(\gamma\phi(\mathcal{H})t+\varphi_0\right)\right), \tag{6}$$

$$p(t) = -\frac{1}{f(x)}\sqrt{2\left(\mathcal{E}-\epsilon\right)m(x)}\sin\left(\gamma\phi(\mathcal{H})t+\varphi_0\right). \tag{7}$$

As an example, let us consider the harmonic oscillator of frequency ω. One can find that for this simple system $f(x) = 1$, $\phi(\mathcal{H}) = 1$ and $\gamma = \omega$, leading to

$$\mathcal{W}(x) = \sqrt{\frac{m_0\omega^2}{2}}\int J(x)dx, \qquad \mathcal{V}(x) = \frac{m_0\omega^2}{2}\left(\int J(x)dx\right)^2 + \epsilon \tag{8}$$

with $J(x) = \sqrt{m(x)/m_0}$ and m_0 a constant with dimensions of mass. Hence, under the transformation

$$\mathcal{P}(x,p) = p/J(x), \qquad \mathcal{X}(x) = \int J(x)dx \tag{9}$$

the Hamiltonian takes the form of a CM harmonic oscillator of position $\mathcal{X}$ and momentum $\mathcal{P}$. Note, however, that for some choices of $m(x)$ the transformation (9) may not map $\mathcal{D}(\mathcal{H})$ onto the whole real line as required if $\mathcal{X}$ should represent the position of the CM oscillator [16], meaning that there are important differences between PDM and CM problems for those cases. Below, we will consider two mass functions in order to illustrate this approach.

In the first place consider the regular mass m_1 leading to the potential $\mathcal{V}_1$

$$m_1(x) = \frac{m_0}{1+(kx)^2}, \qquad \mathcal{V}_1(x) = \frac{m_0\omega^2}{2k^2}\text{arcsinh}^2 kx \tag{10}$$

with k a constant in inverse position units (observe that the case of constant mass is recovered in the limit $k \to 0$). In this case we have

$$x_1(t) = \frac{1}{k}\sinh\left[\sqrt{\frac{2(\mathcal{E}-\epsilon)}{m_0}}\frac{k}{\omega}\cos\left(\omega t+\varphi_0\right)\right] \tag{11}$$

$$p_1(t) - = \sqrt{\frac{2m_0\left(\mathcal{E}-\epsilon\right)}{1+(kx(t))^2}}\sin\left(\omega t+\varphi_0\right). \tag{12}$$

Figure 1 shows the potential $\mathcal{V}_1$ and some phase trajectories for different values of the total energy of the system. One can note that they are soft deformations of that of the CM oscillator, with the position and momentum taking, in principle, arbitrary values.

Next, we consider the singular mass m_2 with potential $\mathcal{V}_2$

$$m_2(x) = \frac{m_0}{(kx)^2}, \qquad \mathcal{V}_2(x) = \frac{m_0\omega^2}{2k^2}\ln^2 kx \tag{13}$$

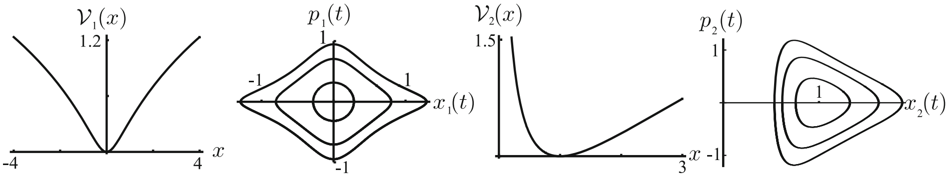

FIGURE 1. The potential and phase space trajectories for m_1 with $\mathcal{E} = 0.1, 0.3, 0.5$ and m_2 for $\mathcal{E} = 0.1, 0.2, 0.3$. In these graphics $m_0 = 2$, $k = 2$, $\omega = 0.8$, $\epsilon = 0.5$ and $\phi = \pi$. Inner curves correspond to lower energies.

for which

$$x_2(t) = \frac{1}{k} \exp\left[\sqrt{\frac{2(\mathcal{E} - \epsilon)}{m_0}} \frac{k}{\omega} \cos(\omega t + \varphi_0)\right] \tag{14}$$

$$p_2(x) = -\frac{\sqrt{2m_0(\mathcal{E} - \epsilon)}}{kx(t)} \sin(\omega t + \phi_0)\,. \tag{15}$$

Figure 1 shows the potential and phase trajectories for m_2. In contrast to the previous case, it is evident the presence of a singularity, confining the motion of the system to a region given by the domain of definition of $m(x)$. It is worthwhile to mention that, even the unusual form of the mass, the behavior of the phase space variables is quite regular. The presence of a divergence in the mass function appears as a potential barrier suggesting that one can define oscillators in bounded domains by introducing masses with singularities.

3. Quantum position-dependent mass systems

In the quantum mechanical regime, it is well known that the canonical variables x, p do not commute and an ambiguity ordering appears in expressions containing products of these variables. A general hermitian Hamiltonian in this case can be defined as

$$H_a = \frac{1}{2} m^a\, p\, m^{2b}\, p\, m^a + V_a(x), \qquad a + b = -\frac{1}{2}, \tag{16}$$

with a the ordering parameter ($b = -a - 1/2$). As mentioned before, the choice of this parameter has been addressed in several ways [8, 16, 19]. In this work it is kept arbitrary, with no more assumptions on a particular ordering of p and m. Similar to the classical case, the form of the potential is found from the algebraic structure underlying the system. Therefore, the eigenvalue equation

$$H_a \psi(x) = E\psi(x) \tag{17}$$

for which the spectrum is well known, can be studied by means of the factorization method.

Suppose then that H_a can be factorized in terms of two linear operators

$$A_a^+ = -\frac{i}{\sqrt{2}} m^a\, p\, m^b + W_a(x), \qquad A_a^- = \frac{i}{\sqrt{2}} m^b\, p\, m^a + W_a(x) \tag{18}$$

in the form

$$H_a = A_a^+ A_a^- + \epsilon. \tag{19}$$

In the position representation $p = -i\hbar d/dx$; hence, defining the differential operator

$$\mathbf{D} = \frac{1}{\sqrt{m(x)}} \frac{d}{dx}, \tag{20}$$

one may write

$$A_a^+ = -\frac{\hbar}{\sqrt{2}} \mathbf{D} + \sqrt{2}\hbar \left(a + \frac{1}{2} \right) \mathbf{D} \left(\ln J(x) \right) + W_a(x) \tag{21}$$

$$A_a^- = \frac{\hbar}{\sqrt{2}} \mathbf{D} + \sqrt{2}\hbar a \mathbf{D} \left(\ln J(x) \right) + W_a(x). \tag{22}$$

It is not difficult to show that the function $W_a(x)$ must satisfy the Riccati equation

$$-\frac{\hbar}{\sqrt{2}} \mathbf{D} W_a + 2\sqrt{2}\hbar \left(a + \frac{1}{4} \right) \left(\mathbf{D} \ln J \right) W_a + W_a^2 = V_a - \epsilon \tag{23}$$

while

$$\left[A_a^-, A_a^+ \right] = \sqrt{2}\hbar \mathbf{D} W_a + 2\hbar^2 \left(a + \frac{1}{4} \right) \mathbf{D}^2 \ln J. \tag{24}$$

For the case in which the factorizing operators close the harmonic oscillator algebra, *i.e.*, $[A_a^-, A_a^+] = \hbar\omega$, we have

$$W_a(x) = \sqrt{\frac{m_0 \omega^2}{2}} \int J(x) dx - \sqrt{2}\hbar \left(a + \frac{1}{4} \right) \mathbf{D} \ln J(x), \tag{25}$$

fixing $V_a(x)$ as

$$\begin{aligned} V_a(x) = \frac{m_0 \omega^2}{2} \left(\int J(x) dx \right)^2 + \hbar^2 \left(a + \frac{1}{4} \right) \mathbf{D}^2 \ln J(x) \\ - 2\hbar^2 \left(a + \frac{1}{4} \right)^2 \left(\mathbf{D} \ln J(x) \right)^2, \end{aligned} \tag{26}$$

which is isospectral to the CM harmonic oscillator: $Sp(H_a) = \left\{ E_n = \left(n + \frac{1}{2} \right) \hbar\omega \right\}$, and lead to wave functions $\psi_n(x)$ given by

$$\psi_n(x) = \frac{1}{\sqrt{n!}} \left(A_a^+ \right)^n \psi_0(x) \tag{27}$$

where $\psi_0(x)$ is the ground state defined by $A_a^- \psi_0(x) = 0$.

At this point, it is important to stress that the subscript a in V_a distinguishes different potentials for different orderings of the kinetic term. However, the Hamiltonian H_a is the same for any value of a, and the subscript only labels different orderings of p and m. Therefore, neither the spectrum, nor the eigenfunctions of H_a should depend on a. Indeed, the substitution of (25) into (18) gives

$$A_a^{\pm} = A^{\pm} = \mp \frac{\hbar}{\sqrt{2}} \mathbf{D} \pm \frac{\hbar}{2\sqrt{2}} (\mathbf{D} \ln J) + \sqrt{\frac{m_0 \omega^2}{2}} \int J dx \tag{28}$$

which are actually independent of the ordering parameter (see [17]). Note also that

$$A^{\pm} J^{1/2} = J^{1/2} \left(\mp \frac{\hbar}{\sqrt{2}} \mathbf{D} + \sqrt{\frac{m_0 \omega^2}{2}} \int J dx \right) = J^{1/2} \mathrm{a}^{\pm}, \tag{29}$$

where we can identify to $\mathrm{a}^{\pm}$ as the ladder operators of the CM harmonic oscillator by making the correspondence $\int J dx \to y(x)$, $\sqrt{m_0}\,\mathbf{D} \to \frac{d}{dy}$.

In this way, if $\psi_0(x) = \sqrt{J(x)} \phi_0(y(x))$, then $\phi_0(y)$ must satisfy

$$\left(\frac{\hbar}{\sqrt{2m_0}} \frac{d}{dy} + \sqrt{\frac{m_0 \omega^2}{2}} y \right) \phi_0(y) = 0 \tag{30}$$

which is nothing but the equation defining the ground state of the CM harmonic oscillator. The whole set of wave functions $\psi_n(x)$ are hence constructed as

$$\psi_n(x) = J^{1/2}(x) \phi_n \left(\int J(x) dx \right), \tag{31}$$

with $\phi_n(y)$ the wave functions of the constant mass harmonic oscillator, consistently with the point canonical transformation [9]. Some plots of potential and corresponding wave functions are presented in Figure 2.

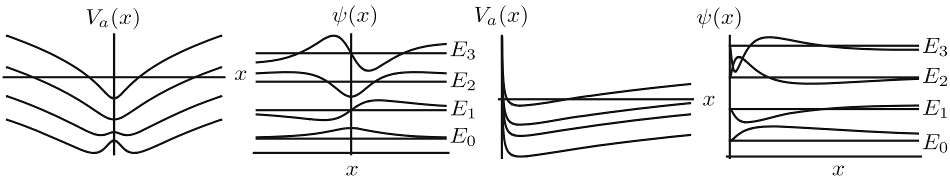

Figure 2. Position-dependent mass potentials and its corresponding first 4 wave functions for masses m_1 (left) and m_2 (right). Observe that the potentials depend on the ordering parameter a, upper curves correspond to smaller values of a. Note though, that the wave functions are the same for any value of a. Here we have used $m_0 = 2$, $k = 2$, $\omega = 0.8$ and $a = 0, 0.25, 0.35, 0.5$.

Observe that the PDM harmonic oscillator Hamiltonian H_a can be also factorized as

$$H_a = A^- A^+ - \frac{\hbar \omega}{2}. \tag{32}$$

It is well known, for the CM case, that the operators $A^{\pm}$ are not unique [20]. It is not difficult to prove this fact also for the PDM potentials, indeed, $W_a(x)$ fulfills the Riccati equation

$$\frac{\hbar}{\sqrt{2}}\mathbf{D}W_a+2\sqrt{2}\hbar\left(a+\frac{1}{4}\right)(\mathbf{D}\ln J)W_a+W_a^2=V_a-2\hbar^2\left(a+\frac{1}{4}\right)\mathbf{D}^2\ln J+\frac{\hbar\omega}{2} \quad (33)$$

with the general solution

$$W_a(x,\Gamma)=\sqrt{\frac{m_0\omega^2}{2}}\int J(x)dx-\sqrt{2}\hbar\left(a+\frac{1}{4}\right)\mathbf{D}\ln J(x) + \frac{\hbar^2}{\sqrt{2}}\mathbf{D}\ln\left[\Gamma+\sqrt{\frac{m_0\omega}{\hbar}}\int_0^{\int J dx} e^{-\frac{m_0\omega}{\hbar}t^2}dt\right], \quad (34)$$

leading to new (a-independent) operators

$$B^{\pm}=A^{\pm}+\frac{\hbar^2}{\sqrt{2}}\mathbf{D}\ln\left[\Gamma+\sqrt{\frac{m_0\omega}{\hbar}}\int_0^{\int J dx} e^{-\frac{m_0\omega}{\hbar}t^2}dt\right] \quad (35)$$

such that $H_a = B^-B^+ - \hbar\omega/2$. It is clear that these operators do not close the Heisenberg algebra, meaning that we can construct new Hamiltonians $\tilde{H}_a$ by applying a Darboux transformation [20]

$$\tilde{H}_a(\Gamma)=B^+B^-+\frac{\hbar\omega}{2}=\frac{1}{2}m^a p m^{2b} p m^a+\tilde{V}_a(x,\Gamma) \quad (36)$$

with

$$\tilde{V}_a(x,\Gamma)=V_a(x)-\hbar^2\mathbf{D}^2\ln\left[\Gamma+\sqrt{\frac{m_0\omega}{\hbar}}\int_0^{\int J dx} e^{-\frac{m_0\omega}{\hbar}t^2}dt\right], \quad (37)$$

which is non singular whenever $|\Gamma| > \frac{\sqrt{\pi}}{2}$. Some plots for the new potentials are shown below.

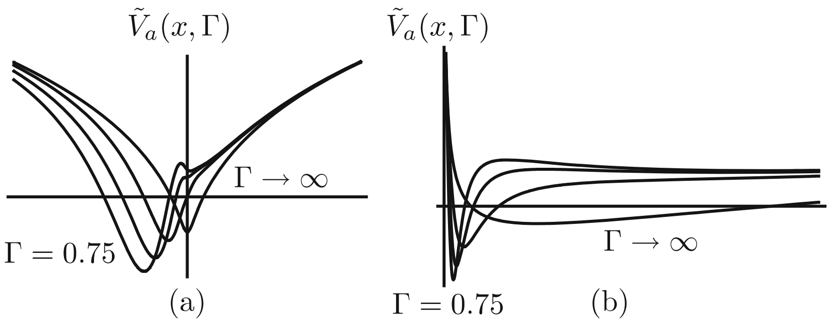

FIGURE 3. Some new PDM potentials isospectral to the harmonic oscillator for different choices of the new parameter Γ. Plots on (a) correspond to m_1 while those in (b) to m_2. In this graphics $m_0 = 2$, $k = 2$,$\omega = 0.8$, $a = 0$, $\Gamma = 0.75, 0.8, 1$ and $\Gamma \to \infty$.

Additionally, both Hamiltonians show the intertwining relations $H_a B^- = B^- \tilde{H}_a$, $B^+ H_a = \tilde{H}_a B^+$, and the wave functions $\theta_n(x)$ of $\tilde{H}_a$ can be easily constructed by the application of $B^\pm$ on the wave functions of H_a:

$$\theta_n(x) = B^+ \psi_{n-1}(x), \qquad n = 1, 2, 3, \ldots \tag{38}$$

corresponding to the spectral values E_n. There is, though, an isolated eigenvector $\theta_0(x)$ of $\tilde{H}_a$, orthonormal to the whole set $\{\theta_n(x), n = 1, 2, \ldots\}$, but not connected to $\{\psi_n(x), n = 0, 1, 2, \ldots\}$ by $B^\pm$ defined as

$$B^- \theta_0(x) = 0, \tag{39}$$

and corresponding to the eigenvalue E_0 [20].

4. Concluding remarks

We have considered the PDM harmonic oscillator from classical and quantum mechanical points of view. In both cases the problem was addressed by means of the factorization method. The technique is consistent with the point canonical transformation. Some examples were presented in order to show the effect of a regular and singular variable mass in the dynamics of the system. In the quantum case, the solution was given for a generalized ordering between m and p. New potentials, isospectral to the CM harmonic oscillators, were obtained from the intertwining relations. The factorization method can be also generalized for different underlying algebraic structure of both, classical and quantum PDM problems [23]. In the quantum case, new PDM supersymmetric partners can be also defined [22,24], and different families of PDM coherent states can be constructed [25]. Results of these generalizations can be found elsewhere [26].

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Quantum Configuration Spaces of Extended Objects, Diffeomorphism Group Representations and Exotic Statistics

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and dedicated also to my colleagues
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Abstract. A fundamental approach to quantum mechanics is based on the unitary representations of the group of diffeomorphisms of physical space (and correspondingly, self-adjoint representations of a local current algebra). From these, various classes of quantum configuration spaces arise naturally, as well as the usual exchange statistics for point particles in spatial dimensions $d \geq 3$, induced by representations of the symmetric group. For $d = 2$, this approach led to an early prediction of intermediate or "anyon" statistics induced by unitary representations of the braid group. I review these ideas, and discuss briefly some analogous possibilities for infinite-dimensional configuration spaces, including anyonic statistics for extended objects in three-dimensional space.

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1. Introduction

It is remarkable how slowly physicists gained insight into exotic possibilities for the statistics of quantum particles. Bose-Einstein and Fermi-Dirac statistics, cor-

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responding respectively to the trivial and alternating one-dimensional representations of the symmetric group S_N, have been known of course since the 1920s. During the 1950s and 1960s, quantum theories were studied obeying "parastatistics," associated with various families of higher-dimensional representations of S_N [1, 2]. During this period, Aharonov and Bohm drew attention to what can now be understand as topological effects in quantum mechanics, associated with charged particles circling (but not entering) regions of magnetic flux [3]. In 1971, Laidlaw and DeWitt explicitly connected the topology of N-particle configuration spaces in $\mathbf{R}^3$ with the familiar possibilities of Bose and Fermi statistics [4]. But the first clear suggestion of the possibility of intermediate statistics for indistinguishable particles in $\mathbf{R}^2$ did not come until a 1977 paper by Leinaas and Myrheim [5], fully half a century after the exchange statistics of bosons and fermions had become standard in quantum mechanics – even though the idea can be obtained and expressed in elementary ways.

An early, independent prediction of such intermediate statistics in the plane came from the study by Menikoff, Sharp, and myself of representations of a certain local current algebra for quantum mechanics, and the associated infinite-dimensional group [6, 7]. This group is the natural semidirect product of the additive group $\mathcal{D} = C_0^\infty(M)$ of compactly-supported, real-valued C^∞ scalar functions on the spatial manifold M, with the group $\mathcal{K} = \mathrm{Diff}_0(M)$ of compactly-supported C^∞ diffeomorphisms of M under composition (where, in the case at hand, $M = \mathbf{R}^2$). Particles satisfying intermediate statistics were subsequently termed "anyons" by Wilczek [8, 9], as wave functions can be multiplied by a fixed complex number of modulus one – $\exp i\theta$, for "any" phase $0 \leq \theta < 2\pi$ – as a consequence of the exchange of indistinguishable particles through a single counterclockwise winding in the plane.

Anyons are associated with the equivariance of wave functions under one-dimensional representations of the braid group B_N [10, 11]. Their description fits nicely into the framework of braided tensor products developed by Majid, and when $\exp i\theta$ is a root of unity, generalized exclusion principles occur [12]. Higher-dimensional braid group representations likewise describe possible quantum particle systems in two-dimensional space [13]; such particles or excitations have been termed "nonabelian anyons" or "plektons."

These ideas have found numerous applications in physics, ranging from the theory of the quantum Hall effect to high-T_c superconductivity to quantum computing; for a recent, extensive discussion focusing on the latter, see Nayak *et al.* [14].

Recently attention has been drawn to possibilities for exotic statistics associated with configurations of extended objects. For example, Niemi discusses anyonic statistics that can occur for "leapfrogging" vortex rings, deriving this possibility in an elementary way that suggests to Niemi that it is generic [15], and providing inspiration for the present discussion. Here, I hope to indicate how such possibilities for the exotic statistics of extended objects arise naturally from the diffeomorphsim group approach to quantum mechanics. With relatively few equations, I shall survey some of the key ideas in this approach, unifying in a way the discussion of extended configuration spaces with that of exotic statistics. More detail about some of these ideas may be found in the references [16, 17, 18],

Section 2 offers a general description of representations of the semidirect product group $\mathcal{D} \times \mathcal{K}$ modeled on various classes of configuration spaces. Section 3 highlights induced representations and corresponding 1-cocycles in the N-particle case. Finally Section 4 indicates briefly how these ideas are generalized to extended objects, including configurations of loops and tori. Possible applications are to those domains of quantum physics where topologically nontrivial objects are fundamental, such as loops, ribbons, or rings of vorticity, configurations of magnetic flux, quantized strings, geons, and so forth.

2. Diffeomorphism group representations and quantum configuration spaces

Let M be the manifold of physical space (assumed to be smooth, connected, separable, locally compact, and σ-compact), and let $\mathbf{x}$ denote a general point in M. The support of a diffeomorphism $\phi : M \to M$ is defined to be the intersection of all closed sets outside of which $\phi(\mathbf{x}) \equiv \mathbf{x}$. The set of compactly supported diffeomorphisms $\mathcal{K}$ of M forms a group under composition: to be precise, we define $\phi_1\phi_2 = \phi_2 \circ \phi_1$, where $\circ$ denotes composition. Then $\mathcal{K}$ is an infinite-dimensional topological group in the topology of uniform convergence in all derivatives on compact sets. Similarly, $\mathcal{D}$ is an infinite-dimensional topological group under addition, endowed with the topology of uniform convergence in all derivatives on compact sets. The semidirect product $G = \mathcal{D} \times \mathcal{K}$ is defined by the group law

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1)\,. \tag{1}$$

In an important sense, G may be considered a fundamental symmetry group of physical space for the purpose of defining the kinematics of quantum mechanics. It is a *local* symmetry group, in that given any fixed compact region $K \subset M$, we have a closed subgroup $\mathcal{D}_K \subset \mathcal{D}$ of functions supported in K (i.e., vanishing outside K), a closed subgroup $\mathcal{K}_K$ of diffeomorphisms having support in K, and the semidirect product $G_K = \mathcal{D}_K \times \mathcal{K}_K$ which is a closed subgroup of $G = \mathcal{D} \times \mathcal{K}$.

The group G is obtained by exponentiating the singular local current algebra proposed in 1968 by Dashen and Sharp [19], interpreted as a Lie algebra of operator-valued distributions [20]. This algebra, in turn, can be obtained formally from canonical creation and annihilation fields. The inequivalent, continuous unitary representations of G then correspond to distinct quantum systems, infinite as well as finite, so that their classification and interpretation becomes of central physical interest [21, 22]. A consequence is that one can describe – and, in fact, predict – exotic particle statistics as well as topological quantum effects, in a mathematically satisfying way. Let us see briefly how this works.

Let $f \in \mathcal{D}$ and $\phi \in \mathcal{K}$, for a particular spatial manifold M. A very general unitary representation of the semidirect product is given by the equations,

$$(\gamma) = \exp i\langle \gamma, f\rangle \Psi(\gamma) \quad \text{a.e.}\,(\mu)\,,$$
$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)} \quad \text{a.e.}\,(\mu)\,, \tag{2}$$

which we shall now spend a little time interpreting and discussing.

In (2), the variable γ ranges over elements of a *quantum configuration space* Δ that one has defined (see below). The first equation requires that we have identified a sense in which γ also acts as a *continuous real-valued linear functional* on $\mathcal{D}$ (i.e., as a distribution over $\mathcal{D}$). The value of the distribution γ at $f \in \mathcal{D}$ is denoted $\langle \gamma, f\rangle \in \mathbf{R}$. That is, the elements of Δ are somehow (see below) identified with some of the elements of the dual space $\mathcal{D}'$. The second equation presupposes a natural, μ-measurable group action of the diffeomorphism group $\mathcal{K} = \mathrm{Diff}_0(M)$ on Δ, denoted by $(\phi, \gamma) \to \phi\gamma$, where μ is a measure on Δ having the important technical property of *quasiinvariance* under this group action. To be precise, this is actually a *right* group action, so that $[\phi_1\phi_2]\gamma = \phi_2(\phi_1\gamma)$. Quasiinvariance means that for all $\phi \in G$, the transformed measure μ_ϕ is absolutely continuous with respect to μ. This implies the existence of the Radon-Nikodym derivative $[d\mu_\phi/d\mu](\gamma)$ *almost everywhere* (a.e.) – i.e., outside of μ-measure zero sets.

Of course, to have such a measure μ, Δ must be a measurable space, endowed with a σ-algebra $\mathcal{B}_\Delta$ of "measurable" subsets which is closed under countable unions, countable intersections, and complements. We shall also need $\langle \gamma, f\rangle$ to be a measurable function of γ, for all $f \in \mathcal{D}$.

Now, in both equations (2), Ψ belongs to a Hilbert space $\mathcal{H}$, denoted $L^2_{d\mu}(\Delta, \mathcal{W})$, and defined to be the space of μ-measurable functions $\Psi(\gamma)$ on Δ, square-integrable with respect to μ, taking values in a complex inner product space $\mathcal{W}$. The inner product in $\mathcal{H}$ is given by

$$(\Phi, \Psi) = \int_\Delta \langle \Phi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}\, d\mu(\gamma)\,, \tag{3}$$

where $\langle\ ,\ \rangle_{\mathcal{W}}$ denotes the inner product in $\mathcal{W}$. When $\mathcal{W} = \mathbb{C}$ (the complex numbers), equation (3) becomes simply $(\Phi, \Psi) = \int_\Delta \overline{\Phi(\gamma)}\Psi(\gamma)\, d\mu(\gamma)$; but when $\mathcal{W}$ is a higher-dimensional space, Ψ may have (finitely or infinitely many) components.

Finally, χ is a measurable, unitary 1-cocycle. This means that (for each $\phi \in \mathcal{K}$) χ_ϕ is a measurable function of $\gamma \in \Delta$ taking values in the group of unitary operators on $\mathcal{W}$; and, furthermore, satisfying for each $\phi_1, \phi_2 \in \mathcal{K}$ the cocycle equation,

$$\chi_{\phi_1\phi_2}(\gamma) \;=\; \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) \;\; \text{a.e.}\,(\mu)\,. \tag{4}$$

Note that the system of Radon-Nikodym derivatives $\alpha_\phi(\gamma) = [d\mu_\phi/d\mu](\gamma)$ is a measurable, positive real-valued cocycle, as is also $\alpha^{1/2}$. Let us remark that the failure sets for cocycle equations here may actually depend on ϕ_1 and ϕ_2 in such fashion that there is no measure zero set outside of which the equation holds for every pair of diffeomorphisms. The factor $\alpha^{1/2}$ in equation (2) is precisely what is needed to ensure that the representation is unitary; indeed, the fact that $V(\phi)$ preserves the inner product in $\mathcal{H}$ is demonstrated simply by making a change of variable in calculating $(V(\phi)\Phi, V(\phi)\Psi)$ using equations (2) and (3). Furthermore the action of the cocycle χ_ϕ in equation(2), being unitary in $\mathcal{W}$, does not alter the value of this inner product.

Unitarily inequivalent representations of G are now to be associated with inequivalent measures μ, and (for equivalent measures) with inequivalent (noncohomologous) cocycles χ.

The representation theory of the diffeomorphism group specified by the second equation in (2), viewed in this way, thus incorporates and unifies two features: (1) the class of possible quantum configuration spaces Δ equipped with quasiinvariant measures, describing the kinds of configurations for which there exists a consistent quantum theory on M (i.e., a consistent quantization of some classical motion in M), and (2) the 1-cocycles with respect to the action of the group $\mathrm{Diff}_0(M)$ on Δ, describing the possible quantum statistics of such configurations (in the generalized sense of statistics that includes exotic statistics).

Let us close this section by mentioning briefly some of the approaches to constructing configuration spaces that are pertinent to this description. More discussion of some of these may be found in earlier papers and the references therein [18, 23].

1. Systems of N *indistinguishable point particles* in M correspond to configuration spaces $\Gamma^{(N)}$ of finite (N-element) subsets of M. When M is noncompact, systems of *infinitely many* such point particles are described by configurations which are countably infinite but locally finite subsets of M, defining the space $\Gamma^{(\infty)}$. When $M = \mathbf{R}^d$, this is the usual configuration space for statistical mechanics [24, 25, 26, 27]. Of course, diffeomorphisms of M act on subsets of M in the obvious way; they do not create or destroy particles, but move them around in M.

2. General configuration spaces may be defined as orbits or unions of orbits (under the diffeomorphism group action) in the space $\mathcal{D}'$ of distributions over M (for $M = \mathbf{R}^d$, one also has the possibility of considering tempered distributions). Particle configurations, in particular, are associated with linear combinations of evaluation functionals (δ-functions) in this space. Coefficients of δ-functions may be interpreted as particle masses, allowing configurations of distinguishable as well as indistinguishable particles to be described in this way. Here diffeomorphisms of M act on $\mathcal{D}$ as specified by the semidirect product law in G, and on distributions by the dual action [20].

3. Letting N be a manifold (typically of lower dimension than M), a class of configuration spaces may be constructed as spaces of (not necessarily infinitely differentiable) embeddings (or, more generally, immersions) of N in M; let us write such a configuration as $\beta : N \to M$. For example, with $N = S^1$, we have configuration spaces of loops in M.

Such embeddings or immersions may be parametrized (so that the map β itself is the configuration), or unparametrized (so that the image set $[\beta]$ of β is the configuration; then $\beta_1 \sim \beta_2$ if they are related by a diffeomorphism of N). For $N = S^1$, we thus have the possibility of parametrized or unparametrized loops. If M is three-dimensional, we also have distinct configuration spaces for different kinds of knots. A prerequisite for the existence of measures on such spaces that are quasiinvariant under (C^∞) diffeomorphisms of M seems to be that the continuity class of β be fixed at a finite value. To the best of my knowledge, this theory is still incomplete.

4. General configuration spaces may be defined as spaces of *closed* subsets of M, as proposed and developed by Ismagilov; see [28] and references therein. Note that unparametrized embeddings or immersions of N in M are special cases of such closed subsets, while parametrized embeddings or immersions are not.
5. Still more general configuration spaces may be defined as spaces of *countable* subsets of M (without imposing the condition of local finiteness). This generalizes $\Gamma^{(\infty)}$, in that it allows for infinite-point configurations with accumulation points. It also generalizes Ismagilov's approach, in that (M being separable) a closed subset can be recovered as the closure of many distinct countable subsets (see [29] and references therein). Parametrized configurations require consideration of ordered countable subsets.
6. Consideration of the *coadjoint representation* of $\mathcal{K}$, or of the semidirect product group $G = \mathcal{D} \times \mathcal{K}$, suggests that one construct configuration spaces from the dual space to the corresponding (infinite-dimensional) Lie algebra – i.e., the dual space to the current algebra of compactly-supported scalar functions and vector fields on M. Then one needs to introduce a "polarization" (in the spirit of geometric quantization) in the corresponding coadjoint orbit or class of orbits, which amounts to selecting certain coordinates as "position-like" and others as "momentum-like" – with the former defining the quantum configurations. The additional (symplectic) structure on coadjoint orbits provides a systematic way to obtain cocycles in this context.
7. Finite or countably infinite subsets of *bundles* over M provide another approach to configuration spaces. For example, returning to configuration spaces in $\mathcal{D}'$, derivatives of δ-functions (including higher derivatives) are perfectly satisfactory configurations, and lead to quantum theories of point-like dipoles, quadrupoles, etc. [30]. However, these configurations belong not to M itself, but to the jet bundle over M, to which the action of diffeomorphisms on M lifts naturally.
8. Finally, in the spirit of the approach via bundles over M, there is a physically important generalization to what has been termed "marked configuration spaces." Here one identifies a compact manifold S describing the "internal degrees of freedom" of a particle, and a compact Lie group L that acts on S. One then associates to each point in an ordinary configuration a value or

"mark" in S [31, 32]. The local symmetry group itself can be correspondingly enlarged to include compactly supported C^∞ mappings from M to L under the pointwise Lie group operation, and/or to include bundle diffeomorphisms of $M \times S$.

Each of these methods of characterizing quantum configuration spaces has some significant literature that develops it, and in some instances is associated with a point of view about quantization or about quantum mechanics. The diffeomorphism group approach helps us understand these distinct but overlapping methods as techniques for the construction of classes of unitary group representations embodying the local symmetry of physical space in the quantum kinematics.

3. Induced representations and particle statistics

Next let us consider briefly the examples of Bose statistics, Fermi statistics, and parastatistics for N indistinguishable particles in $\mathbf{R}^d$, $d \geq 2$, and of anyonic statistics for N (distinguishable or) indistinguishable particles in $\mathbf{R}^2$.

The configuration space $\Gamma^{(N)}$ is the set of N-point subsets of $\mathbf{R}^d$; we write $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \in \Gamma^{(N)}$. The space $\Gamma^{(N)}$ is sometimes written in the more complicated way $[\mathbf{R}^{dN}\text{-}D]/S_N$; where $\mathbf{R}^{dN}$ is the set of ordered N-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of points in $\mathbf{R}^d$, D is the "diagonal" set of N-tuples for which $\mathbf{x}_i = \mathbf{x}_j$ for some $i \neq j$, and S_N is the symmetric group for N objects. Thus $\Gamma^{(N)}$ is the set of ordered N-tuples without repeated points, modulo all permutations of the values of the points. A diffeomorphism ϕ acts on $\Gamma^{(N)}$ by (the right action) $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \to \phi\gamma = \{\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_N)\}$.

Note that for $d \geq 2$, $\Gamma^{(N)}$ is *multiply connected* – indeed, any continuous path in $\Gamma^{(N)}$ that begins at a configuration γ_0 and non trivially *permutes* the locations of the points in γ_0 forms a closed loop in the configuration space, based at γ_0, that cannot be continuously contracted to γ_0.

First let us consider $d \geq 3$. The fundamental group $\pi_1(\Gamma^{(N)})$, which is the group of distinct homotopy classes of such loops (under composition), is then just isomorphic to S_N, according to the particular permutation of the locations of the points in γ_0 implemented by a loop based there. The *universal covering space* $\tilde{\Gamma}^{(N)}$ is then the space of *ordered* N-tuples without repeating points; i.e., $\tilde{\Gamma}^{(N)} = [\mathbf{R}^{dN}\text{-}D]$, and we shall write $\tilde{\gamma} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \tilde{\Gamma}^{(N)}$. Then we have the projection $p : \tilde{\Gamma}^{(N)} \to \Gamma^{(N)}$ from the universal covering space to the base space, given by $p(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$; i.e., p tells us to "forget the ordering." There are, of course, $N!$ sheets in $\tilde{\Gamma}^{(N)}$ (for $d \geq 3$), corresponding to the $N!$ elements of the fundamental group S_N.

Consider next the action of $\mathcal{K} = \mathrm{Diff}_0(\mathbf{R}^d)$ on $\Gamma^{(N)}$. The stability subgroup $\mathcal{K}_\gamma \subset \mathcal{K}$ consists of those compactly-supported diffeomorphisms which leave γ fixed; i.e., just those which permute the points in γ. Thus $\mathcal{K}_\gamma$ contains $N!$ disconnected components, and we obtain a natural homomorphism h_γ from $\mathcal{K}_\gamma$ to S_N. Referring back to equations (2) and (4), observe that when ϕ_1 and ϕ_2 belong to

$\mathcal{K}_\gamma$, the cocycle equation at γ becomes a unitary representation in $\mathcal{W}$ of $\mathcal{K}_\gamma$. Thus we have an association between cocycles describing quantum theories modeled on $\mathbf{R}^d$ $(d \geq 3)$ and unitary representations of $\mathcal{K}_\gamma$. Note too that any unitary representation π of S_N in the inner product space $\mathcal{W}$ now gives us a continuous unitary representation $\pi \circ h_\gamma$ of $\mathcal{K}_\gamma$ in $\mathcal{W}$. Cocycles describing quantum theories of Bose statistics, Fermi statistics, and parastatistics correspond in this way to inequivalent representations of S_N: the trivial (Bose) and alternating (Fermi) one-dimensional representations (for $\mathcal{W} = \mathbb{C}$), and additional (para) higher-dimensional representations described by Young tableaux (with $\mathcal{W} = \mathbb{C}^n$).

The corresponding unitary representations of $\mathcal{D} \times \mathcal{K}$ can be obtained in a different way, making use of a generalization of Mackey's theory of induced representations. The action of $\phi \in \mathcal{K}$ on $\Gamma^{(N)}$ lifts naturally to an action $\tilde{\phi}$ on the universal covering space $\tilde{\Gamma}^{(N)}$, so that $\phi(p\tilde{\gamma}) = p\tilde{\phi}(\tilde{\gamma})$. Diffeomorphisms belonging to $\mathcal{K}_\gamma$, in their action on $\tilde{\Gamma}^{(N)}$, now *permute* the elements of $p^{-1}\gamma$. In the induced representation approach, the Hilbert space consists of wave functions on $\tilde{\Gamma}^{(N)}$ that are *equivariant* with respect to the given unitary representation of the fundamental group S_N, and thus with respect to the corresponding unitary representation of $\mathcal{K}_\gamma$ in its action on $\tilde{\Gamma}^{(N)}$.

In short, for $d \geq 3$, we see how the topology of the N-particle configuration spaces in $\mathbf{R}^d$ gives rise to the possible exchange statistics of indistinguishable particles in the representation theory of the group of diffeomorphisms of $\mathbf{R}^d$. Corresponding to the unitary representations of the fundamental group of Δ are inequivalent cocycles for the action of $\mathrm{Diff}_0(M)$ on Δ, and different equivariance conditions for wave functions written on the universal covering space of Δ.

Finally, consider the case $d = 2$. The fundamental group $\pi_1(\Gamma^{(N)}(\mathbf{R}^2))$ is larger than S_N, because loops based at a configuration γ_0 that implement (let us say) a clockwise exchange of two points of γ_0 in $\mathbf{R}^2$ are not homotopically equivalent to loops that implement a counterclockwise exchange of the same two points. Here, the fundamental group is Artin's braid group B_N, an infinite discrete group which for $N > 2$ is nonabelian. One may think of the braid group element b_j, for $j = 1, \ldots, N-1$, as exchanging the pair of points $\mathbf{x}_j, \mathbf{x}_{j+1}$ (which are *adjacent* with respect to some coordinatization of the plane), in a counterclockwise direction; the element b_j^{-1} then exchanges the same pair of points in a clockwise direction. The group B_N is the free group generated by these elements, *modulo* the equivalence relation $b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}$.

Now the space of ordered N-tuples of points in the plane is a covering space of $\Gamma^{(N)}(\mathbf{R}^2)$, but it is no longer the *universal* covering space; the latter has infinitely many sheets. Ultimately wave functions on the universal covering space, equivariant with respect to a unitary representation of the braid group, define the Hilbert space for the desired representation of G.

We omit further details, but close this section by focusing on a key step in this induced representation construction for anyons, which we shall then indicate how to generalize to configurations of extended objects. This step is the association of the connected components of the stability subgroup $\mathcal{K}_\gamma$ (i.e., the subgroup of compactly supported diffeomorphisms of the plane that leave fixed the subset of points $\gamma = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$) with elements of the fundamental group B_N, by means of a homomorphism h_γ.

One way to define this homomorphism, described in Ref. [33], is as follows. Choose an arbitrary direction in the plane M, let us say for specificity the y-direction with respect to Cartesian coordinate axes x and y, such that for the points in the configuration γ no two x-coordinates coincide. Index the points $\mathbf{x}_j$ in order of increasing x-coordinate value. Attach to each point in γ a *strand* which is a straight line extending to infinity in the negative y-direction; the parallel strands in this set of strands do not intersect. Now a compactly-supported diffeomorphism ϕ in the stability subgroup of γ leaves all of the strands fixed at infinity (because of the compact support of ϕ), but can permute their terminal points. Still more generally, the images of the strands of under ϕ constitute a *new* set of non-intersecting strands coming in from $y = -\infty$ and terminating at the points in γ. This set of strands may be homotopically inequivalent to the original set, *even when* $\phi(\mathbf{x}_j) = \mathbf{x}_j$ *for all* j; i.e., even when ϕ implements no permutation of the points.

In fact, such sets of strands fall into distinct homotopy classes, encoding the passages of strands above or below each other (with respect to the coordinate y) as one moves in from $y = -\infty$ to the points of γ. When no such passage occurs, we map ϕ to the identity element of B_N. When the only such passage is that the strand terminating in $\mathbf{x}_{j+1}$ passes once above the strand terminating in $\mathbf{x}_j$, we map the diffeomorphism to b_j. In this way, the stability subgroup $\mathcal{K}_\gamma$ is mapped homomorphically to B_N.

Then a unitary representation of B_N in a space $\mathcal{W}$ immediately implements a continuous unitary representation of $\mathcal{K}_\gamma$, which induces the desired representation of G. In short, all the needed information about braiding in encoded in the compactly supported diffeomorphism belonging to the stability subgroup. The one-dimensional representations of B_N, in which b_j is represented by $\exp i\theta$, describe anyons; while the higher-dimensional representations describe nonabelian anyons.

One draws certain physical inferences immediately from the above construction.

First, it is not a prerequisite for intermediate statistics in the plane that there be a hard core potential excluding two or more particles from occupying the same position in M, any more than such a potential is required for ordinary Bose or Fermi statistics. Diffeomorphisms act transitively on the configuration space $\Gamma^{(N)}$, and cannot bring distinct points into coincidence. Thus configuration spaces from which the diagonal D is not excluded may be written as the union of mutually disjoint orbits under the group action, and the corresponding possible irreducible unitary representations still include those that are anyonic.

Secondly, it is not a prerequisite for exotic statistics of particles in the plane that they be indistinguishable. The configuration space of *ordered* N-tuples of points in the plane, excluding N-tuples with coincident points, is still multiply-connected. Its fundamental group is the group of "colored braids." Correspondingly, given such a configuration, the elements ϕ of $\mathcal{K}$ for which $\phi(\mathbf{x}_j) = \mathbf{x}_j$ for all j form a closed subgroup. Elements of this subgroup map the original set of parallel strands (from $y = -\infty$, terminating at the points $\mathbf{x}_j$) to various non-homotopic sets of strands from $y = -\infty$ terminating at the same points.

4. Exotic statistics for extended configurations

The ideas in the preceding sections generalize to consideration of topologically nontrivial configurations in higher-dimensional physical spaces. Let us consider just a couple of examples.

Suppose that Δ is a configuration space whose elements are unparametrized single oriented loops in (for specificity) $\mathbf{R}^3$; i.e., a configuration $\gamma \in \Delta$ is a continuous embedding $[\beta]$ of S^1 (modulo C^∞ reparametrization) of some smoothness class, for which (let us say) the arc length in the target space is defined. Diffeomorphisms of $\mathbf{R}^3$ act on Δ in the obvious way. We remark that we shall not be able to concentrate a quasiinvariant measure on a single orbit under $\mathcal{K}$, but will need an uncountable family of orbits. Nevertheless, we envision being able to infer exotic statistics by selecting configurations from such a family of orbits in a measurable way, and describing topological invariants across orbits of the way diffeomorphisms act on such sets of loops.

For a particular oriented loop γ, consider the stability subgroup $\mathcal{K}_\gamma$. An element $\phi \in \mathcal{K}_\gamma$ leaves the loop invariant as a set, but not necessarily pointwise. Thus there is a homomorphism h_γ that maps $\mathcal{K}$ to $\mathrm{Diff}(S^1)$, with $h_\gamma(\phi)$ specified straightforwardly by looking at how ϕ transforms γ (parametrized by its own arc length). A unitary representation of $\mathrm{Diff}(S^1)$ may then describe the "internal statistics" of γ. This is, in a sense, analogous to the ordinary statistics of particles – an equivariance condition for wave functions can be written that depends only on γ and $\phi\gamma$.

But ϕ encodes still more information. If we introduce a set of continuous, non-selfintersecting strands that become parallel (say, for specificity, on the surface of a circular cylinder) in some fixed direction at infinity, and that terminate at correspondingly ordered points of γ, we see that because ϕ is compactly supported, its action on these strands keeps track of how many times it has, in effect, *rotated* the loop. The stability subgroup thus maps not just to $\mathrm{Diff}(S^1)$, but to a *covering group* of $\mathrm{Diff}(S^1)$. "Bringing the loop in from infinity" (and watching what $\phi \in \mathcal{K}_\gamma$ does) tells us how many windings ϕ is to be associated with. Diffeomorphisms that leave every point of γ fixed still encode the number of rotations, and we have the possibility of introducing an extra, additional phase for a single directed rotation of γ. The loop thus can have internal "intermediate statistics."

If instead of a loop γ is an embedded torus (the continuous image of $S^1 \times S^1$) of some smoothness class, the same idea allows us to associate a pair of winding numbers with a compactly-supported diffeomorphism that leave the torus pointwise fixed. Thus we infer further possibilities for intermediate statistics, associating distinct phases with each directed rotation.

Next consider a configuration γ that is the union of a point and an embedded, oriented loop. Imagine a non-selfintersecting cylindrical membrane from infinity (in some fixed direction) that is bounded by the loop, and a strand from infinity in the same direction, intersecting neither itself nor the membrane. Let us say, for specificity, that at infinity the strand is inside the cylinder, terminating at the point particle inside the cylinder. Now, consider the image of this strand-cylinder combination under the action of $\phi \in \mathcal{K}_\gamma$. The image of the strand may now pass through and around the loop as the image of the membrane is moved to one side, so that the strand finally reaches the particle from "outside" the membrane. All such topological complexity takes place within the compact support of ϕ; outside of this support, the original strand and cylinder are fixed. We see from the homotopy class of this image that ϕ encodes the net number of times the particle passes *through* the oriented loop, and again we can have an arbitrary associated phase.

Finally, consider a configuration γ that is the union of a pair of oriented loops in $\mathbf{R}^3$; the discussion will readily extend to pairs of closed filaments of vorticity, vortex rings, or tori. Now we envision two non-intersecting and non-selfintersecting membranes extending to infinity in a fixed direction, bounded by the respective loops. Suppose that a compactly-supported diffeomorphism $\phi \in \mathcal{K}_\gamma$ exchanges the loops. The homotopy class of the pair of image membranes is now labeled by the sequence of passages of one loop through the other. The diffeomorphism encodes "leapfrogging" as a sequence of such passages. The condition of equivariance of the wave function on configuration-space with respect to a unitary representation of $\mathcal{K}_\gamma$ can associate (in particular) a phase with each such passage, leading again to anyonic statistics.

In conclusion, the idea of describing quantum systems by means of continuous unitary representations of the infinite-dimensional group $G = \mathcal{D} \times \mathcal{K}$ leads to a unifying kinematical description of interesting quantum configuration spaces and associated possibilities for exotic statistics.

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Convex Geometry: A Travel to the Limits of Our Knowledge

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Abstract. Our knowledge and ignorance concerning the geometry of quantum states are discussed.

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Questions about the structure

Physical theories are usually created by accumulating some fragments of information which at the beginning do not allow to predict the final structures. The classical mechanics was formulated by Isaac Newton in terms of mass, force, acceleration and the three dynamical laws. It was not immediate to see the Lagrangians, Hamilton equations and the simplectic geometry behind. We cannot guess the reaction of Newton if he were informed that he was just describing the classical phase spaces defined by the simplectic manifolds...

Quite similarly, Max Planck, Niels Bohr, Louis de Broglie, Erwin Schrödinger and Werner Heisenberg could not see from the very beginning that the physical facts which they described would be reduced by Born's statistical interpretation to the Hilbert space geometry (as it seems, neither Hilbert could predict that). Yet, once accepted that the pure states of a quantum system can be represented by vectors of a complex linear space and the expectation values are just quadratic forms, the Hilbert spaces entered irremediably into the quantum theories. Together appeared the "density matrices" as the mathematical tools representing either pure or mixed quantum ensembles. Their role is now so commonly accepted that its origin is somehow lost in some petrified parts of our subconsciousness: an obligatory element of knowledge which the best university students (and the future specialists) learn by heart. However, is it indeed necessary? Can indeed the interference pictures of particle beams limit the fundamental quantum concepts to vectors in linear spaces and "density matrices"?

Quantum logic?

The desire to find some deeper reasons led a group of authors to postulate the existence of an "intrinsic logic" of quantum phenomena, called the *quantum logic* [1, 2, 3]. Generalizing the classical ideas, it was understood as the collection Q of all statements (informations) about a quantum object, possible to check by elementary "yes-no measurements". Following the good traditions, Q should be endowed with *implication* ($\Rightarrow$), and *negation* $a \to a'$. The implication defines the partial order in Q ($a \Rightarrow b$ reinterpreted as $a \leq b$), suggesting the next axioms about the existence of the lowest upper bound $a \vee b$ ("or" of the logic) and the greatest lower bound $a \wedge b$ ("and" of the logic) for any $a, b \in Q$. The "negation" was assumed to be involutive, $a'' \equiv a$, satisfying de Morgan law: $(a \vee b)' \equiv a' \wedge b'$ as well as other axioms granting that Q is an orthocomplemented lattice [1]. Until now, the whole structure looked quite traditional. With one exception: in contrast to the classical measurements, the quantum ones do not commute, which traduces itself into breaking the *distributive law* $(a \vee b) \wedge c \neq (a \wedge c) \vee (b \wedge c)$ obligatory in any classical logic. The quantum logic was non-Boolean! An intense search for an axiom which would generalize the distributive law, admitting both classical and quantum measurements, in agreement with Birkhoff, von Neumann, Finkelstein [1, 2, 3] and thanks to the mathematical studies of Varadarajan [4] convinced C. Piron to propose the *weak modularity* as the unifying law. To some surprise, the subsequent theorems [4, 5] exhibit certain natural completeness: the possible cases of "quantum logic" are exhausted by the Boolean and Hilbertian models, or by combinations of both. As pointed out by many authors this gives the theoretical physicists some reasonable confidence that the formalism they develop (with Hilbert spaces, density matrices, etc.) does not overlook something essential, so there will be no longer need to think too much about abstract foundations.

However, isn't this confidence a bit too scholastic? It can be noticed that the general form of quantum theory, since a long time, is the only element of our knowledge which does not evolve. While the "quantization problem" is formulated for the existing (or hypothetical) objects of increasing dimension and flexibility (loops, strings, gauge fields, submanifolds or pseudo-Riemannian spaces, non-linear gravitons, etc.), the applied quantum structure is always the same rigid Hilbertian sphere or density matrix insensitive to the natural geometry of the "quantized" systems. The danger is that (in spite of all "spin foams") we shall invest a lot of effort to describe the relativistic space-times in terms of perfectly symmetric, "crystalline" forms of Hilbert spaces, like rigid bricks covering a curved highway. Is there any other option?...

Convex geometry

The alternatives arise if one decides to describe the statistical theories in terms of geometrical instead of logical concepts. What is the natural geometry of the statistical theory? It should describe the pure or mixed particle *ensembles* (also ensembles of multiparticle systems, including the mesoscopic or macroscopic objects). Suppose that one is not interested in the total number of the ensemble individuals, but only in their "average properties". Two ensembles with the same statistical averages cannot be distinguished by any statistical experiments: we thus say that they define the same *state*. Now consider the set S of all *states* for certain physical objects. Even in absence of any analytic description, there must exist in S some simple empirical geometry. Given any two states $x_1, x_2 \in S$ (corresponding to certain ensembles $\mathcal{E}_1$, $\mathcal{E}_2$) and two numbers $p_1, p_2 \geq 0$, $p_1 + p_2 = 1$, consider a new ensemble $\mathcal{E}$ formed by choosing randomly the objects of $\mathcal{E}_1$ and $\mathcal{E}_2$ with probabilities p_1 and p_2; its state, denoted $x = p_1 x_1 + p_2 x_2$ is the *mixture* of x_1 and x_2 in proportions p_1, p_2. If in turn both x_1, x_2 are mixtures of $y_1, y_2 \in S$, then some more information is needed to determine the contents of y_1 and y_2 in x. It can be most simply provided by representing S as a subset of a certain *affine space* Γ, so that $p_1 x_1 + p_2 x_2$ becomes a linear combination. For any two points $x_1, x_2 \in S$ all mixtures $p_1 x_1 + p_2 x_2$ ($p_1, p_2 \geq 0$, $p_1 + p_2 = 1$) form then the straight line interval between x_1 and x_2, contained in S. Hence, S is a convex set [6, 7]. To describe the limiting transitions, Γ must possess a topology and S should be closed in Γ.

The information encoded in the convex structure of S might seem poor: it tells only which states are mixtures of which other states (see Figure 1). Yet it turns out that it contains all essential information about both, logical and statistical aspects of quantum theory.

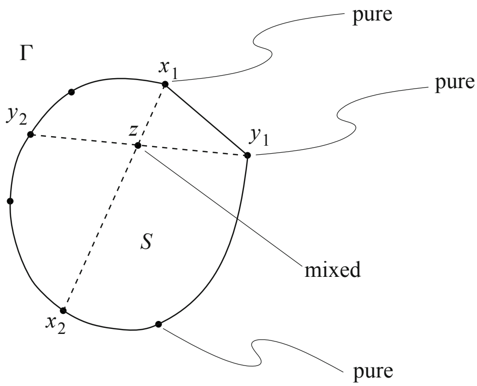

FIGURE 1. A convex set in 2D. Supposing that it could represent the states of some hypothetical ensembles, all border points except the straight line interval joining x_1 and y_1 would represent the pure states. All points in the interior are mixed states and do not allow a unique definition of the pure components. Thus, e.g., the state z could be represented as a mixture of x_1 and x_2 or y_1 and y_2 or in any other way.

Logic of properties

The boundary of S contains some special points x, which *are not* nontrivial combinations $p_1x_1+p_2x_2$ with $p_1,p_2>0$ of any two different points $x_1 \neq x_2$ of S. These points, called *extremal*, represent the physical ensembles which *are not mixtures* of different components, and so are called *pure*. All subensembles of a pure ensemble define the same *pure state* x, which therefore represents also the quality of each single ensemble individual.

The convex geometry permits to describe as well more general ensemble properties which might be attributed to the single individuals. Note that, in general, the property of ensemble is not shared by the individuals (e.g., a human ensemble can contain 50% of men and 50% of women, but each individual, in general, has only one of these qualities). We say that the subset $P \subset S$ defines a *property* of the single objects if: 1. it resists mixing, i.e., $y_1, y_2 \in P$, $p_1, p_2 \geq 0$, $p_1+p_2 = 1 \Rightarrow p_1y_1+p_2y_2 \in P$ (meaning that P is a convex subset of S), 2. if the property of mixture is shared by every mixture components, i.e., $y \in P$, $y = p_1y_1 + p_2y_2$, $y_1, y_2 \in S$, $p_1, p_2 > 0 \Rightarrow y_1, y_2 \in P$. The subset $P \subset S$ which satisfies 1. and 2. is called a *face* of S. The whole S and the empty set $\varnothing$ are the improper faces: all other faces are plane fragments of various dimensionalities on the boundary of S (See Figure 2). In particular, each extremal point of S is a one point face. In what follows, we shall be most interested in the topologically *closed* faces of S representing the "continuous properties" of the ensemble objects. Further on by *faces* we shall mean *closed faces*. Their whole family $\mathcal{P}$ admits a partial ordering $\leq$ identical with the set theoretical inclusion: the relation $P_1 \leq P_2 \Leftrightarrow P_1 \subset P_2$ means that the property P_1 is *more restrictive* than P_2, or P_1 *implies* P_2. As easily seen, the intersection of any family of faces is again a face of S. Hence, for any two faces $P_1, P_2 \subset S$ there exists also their *smallest upper bound*, or *union* $P_1 \vee P_2$, defined as the intersection of all faces containing both P_1 and P_2. The set $\mathcal{P}$ with the partial order $\leq$ (i.e., implication) and operations $\vee$, $\wedge$ is thus a complete lattice generalizing the "quantum logic" of the orthodox quantum mechanics: it might be called the *logic of properties*. Although it does not necessarily include *negation*, but it admits a natural concept of *orthogonality* [6, 8].

Counters

A natural counterpart of quantum ensembles are the measuring devices and the simplest such devices are *particle counters*. By a *counter* we shall understand any macroscopic body sensitive (either perfectly or partly) to the presence of quanta. Our assumption is also, that each counter reacts only to the properties of each single ensemble individual, without depending on the rest. In mathematical terms, each counter can be described by a certain functional $\phi : S \to [0,1]$. whose values ϕx for any $x \in S$ mean the fraction of particles in the state x detected by the counter ϕ. If $\phi x = 1$, then the counter ϕ detects perfectly all x-particles, if $0 < \phi x < 1$, it overlooks a part, but if $\phi x = 0$, then ϕ is completely blind to the x-particles. Moreover, if ϕ reacts only to single ensemble individuals, then

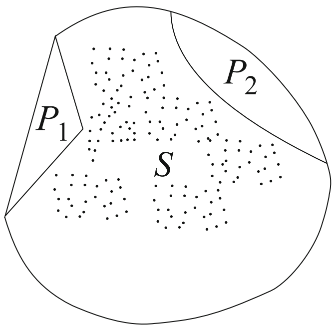

Figure 2. "Faces" on the border of S represent properties of the single ensemble individuals. The picture in perspective permits to see that P_1 and P_2 are not orthogonal.

for any mixed state $x = p_1x_1 + p_2x_2$ it will detect independently both mixture components: $\phi x = p_1\phi x_1 + p_2\phi x_2$, meaning that ϕ is a *linear functional* on S. We shall assume, that the values of counters permit to distinguish the different points $x \in S$ and moreover, they induce a physically meaningful topology, in which they are *eo ipso* continuous. Each continuous, linear functional ϕ taking on S the values $0 \le \phi x \le 1$ will be called *normal*. Mathematically, the counters are, therefore, the *normal functionals*. To get their geometric image, assume that the surrounding affine space $\Gamma \supset S$ is spanned by S. Hence, every linear functional ϕ on S defines a unique linear functional on Γ which will be denoted by the same symbol ϕ. If $\phi \equiv$ const. on S, then $\phi \equiv$ const. on Γ. If not, then the equations $\phi x = c$, $(c \in \mathbb{R})$ split Γ into a continuum of parallel hyperplanes on which ϕ accepts distinct constant values. Due to the linearity, ϕ is completely defined by the pair of hyperplanes on which it takes the values 0 and 1. If ϕ is normal, S is contained in the closed belt of space between both planes. The question arises, how ample is the set of physical counters? Since no restrictions are evident, we shall assume that each normal functional represents a particle counter which at least in principle can be constructed. All distinct ways of counting particles can be thus read from the convex geometry of S [8]. They turn out closely related with the collection of hyperplanes and those are related with *faces*. Indeed, the hyperplanes $\phi = 1$ and $\phi = 0$ of a counter do not cross the interior of S, but can "touch" its boundary. As one can easily show, their common parts with the border ∂S are two "opposite" faces (properties) of S, which awake completely different reactions of the counter: while detecting all particles on one of them, it ignores completely the particles on the other. Any two faces P_1, P_2, for which there exists at least one, so sharply discriminating counter, will be called *excluding* or *orthogonal* $(P_1 \perp P_2)$. The "logic of properties", therefore, is a lattice with the relations of *inclusion* $(\le)$ and *exclusion* $(\perp)$ though without a unique ortho-*complement* (since for any $P \in \mathcal{P}$, amongst all elements orthogonal to P no greatest one must exist).

Detection ratios

Apart from orthogonality, the next geometry element of S describes the selectivity limits of quantum measurements. Given a pair of pure states $x, y \in S$, consider the family of all counters ϕ detecting unmistakeably all particles of the state x, i.e., $\phi x = 1$. Can they ignore completely the particles of the state y? In general, the answer is negative. The following lower bound over the counters ϕ:

$$y : x = \inf_{\phi x = 1} \phi y \tag{1}$$

called the "detection ratio" [8], if non-vanishing, describes a minimal fraction of y-particles which must infiltrate any experiment programmed to detect the x-state. The geometric character of this quantity is defined just by convex structure of S, which determines the support planes (see Figure 3). The information contained

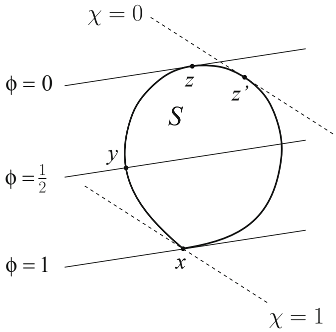

FIGURE 3. The *could be* convex set S for some hypothetical ensembles. The parallel support lines $\chi = 1$ and $\chi = 0$ represent a counter detecting all x-particles, blind to z'-particles, while the lines $\phi = 1$ and $\phi = 0$ correspond to another counter, detecting all x-particles, but the minimal possible fraction $\phi y = \frac{1}{2}$ of the y-particles. Hence, the detection ratio $y : x = \frac{1}{2}$.

in (1) might be significantly weaker if the pure state x were not *exposed*, i.e., determined completely as the intersection of S and at least one support hyperplane. Such cases do not occur in the orthodox QM, but belong to the general convex set geometry (see [9, Fig. 12]).

The orthodox geometry

In the orthodox theory the pure states are represented by vectors in a complex, linear space (an inspiration from the observed interference patterns) and all measured expectation values are real, quadratic forms of the state vectors ψ (the consequence of Born's statistical interpretation). The mixed states are the probability measures on the manifold of pure states (the projective Hilbert sphere). However, since the statistical averages are no more than quadratic forms, the ample classes of probability measures (interpreted as the prescriptions of forming mixtures) are physically indistinguishable. The faithful representation of the mixed states as the equivalence classes of probability measures explains the origin of the “density matrices” [8].

The elements of the convex geometry provide also an alternative interpretation of the unitary invariants $|\langle\psi,\varphi\rangle|^2$ called currently the “transition probabilities”. In fact if S is the convex set of density matrices in a Hilbert space and if two pure states are represented as $x = |\psi\rangle\langle\psi|$ and $y = |\varphi\rangle\langle\varphi|$ ($\|\psi\| = \|\varphi\| = 1$) then the elementary lemma shows that

$$|\varphi\rangle\langle\varphi| : |\psi\rangle\langle\psi| = |\langle\psi,\varphi\rangle|^2 \tag{2}$$

i.e., the commonly used invariant turns out the detection ratio [8], revealing an additional sense of the “transition probabilities”. In fact, as once noticed by Peter Bergman, the deepest picture of a physical theory is obtained not so much by telling what is possible, but rather by “no go principles”, defining what is ruled out (e.g., the equivalence principle in General Relativity, or the uncertainty principle in QM). One such law emerges from the identity (2). Indeed, $|\langle\psi,\varphi\rangle|^2$ not only defines the fraction of the φ-particles accepted by the ψ-filter, but also the fundamental impossibility of accepting less! Every physical process which leads to a certain macroscopic effect for *all* ψ-particles, must lead to the same effect

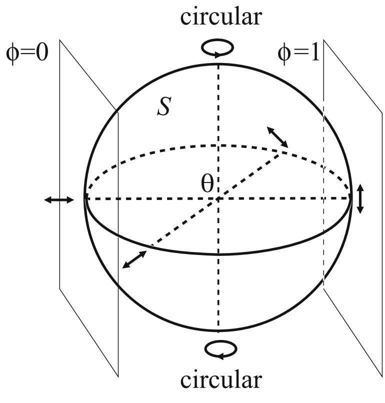

FIGURE 4. Multiple experiments justify the representation of the photon polarization states in form of the 1-qubit (Bloch) sphere in $\mathbb{R}^3$. Once fixed the image, the geometry of S determines uniquely the “transition probabilities” between any pair of pure states. On the figure: the pair of support planes $\phi = 1$ and $\phi = 0$ illustrates the maximally selective counter which detects all photons in the vertical polarization $\updownarrow$, but none in the horizontal $\leftrightarrow$. The intermediate values of ϕ on the congruence of parallel planes intersecting S define the transition probabilities from all other states to the vertical one $\updownarrow$.

at least for the fraction $|\langle\psi,\varphi\rangle|^2$ of φ-particles. The purely geometric nature of this law, independent of any analytic expression can be best illustrated by the Bloch sphere of the photon polarization states (see Figure 4) on which the linear polarizations occupy a great circle (the "equator"), circular polarizations are the poles, and the remaining surface points are the elliptic polarizations. The interior of the sphere collects the mixed polarizations, the center θ representing the complete polarization chaos. The pair of tangent planes $\phi = 1$ and $\phi = 0$ represents a maximally selective counter detecting all photons in the vertical polarization and rejecting the orthogonal one. The geometry of the sphere S determines immediately the "transition probabilities" between any two pure states without the need of using the analytic $|\varphi\rangle\langle\varphi|$ representation (thus, e.g., the detection ratio between any linear and circular polarization is 1/2).

In case of any non-classical ensembles, the geometry of S expresses still more fundamental law about the indistinguishability of quantum mixtures, the phenomenon which appears if S is not a simplex. *Given a mixed ensemble of non-classical objects, one cannot, in general, retrospect and find out how the mixture has been prepared. Two mixtures composed of different collections of pure states can be physically indistinguishable* (*see also Figure* 1).

In the Bloch sphere of polarization states (Figure 4) this effect is exceptionally simple for the center θ which can be represented equivalently as a mixture of any pair of orthogonal linear polarizations, or two opposite circular polarizations or in any other way:

$$\begin{aligned}\theta &\equiv \frac{1}{2} \updownarrow + \frac{1}{2} \leftrightarrow \\ &\equiv \frac{1}{2} \swarrow\!\!\!\nearrow + \frac{1}{2} \nwarrow\!\!\!\searrow \\ &\equiv \frac{1}{2} \circlearrowleft + \frac{1}{2} \circlearrowright \\ &\equiv \cdots\cdots\end{aligned} \tag{3}$$

Hence, once having the mixed state θ one cannot go back and identify its pure components: a kind of statistical no go principle making it quite difficult to check experimentally some semantic curiosities of the existing theory!

Generalized geometries: are they possible?

The structures reported here contain a certain puzzle. It is basically not strange that the convex geometry is a language of statistical theories. Yet, it was not expected that the structure of an arbitrary convex set S contains the equivalents of principal quantum mechanical concepts. Their properties are distorted, but their meaning is similar. Thus, the *logic of properties* is an analogue of the *quantum logic* [1] and the *detection ratios* are equivalents of the orthodox "transition probabilities". In many aspect the Hilbertian schemes are distinguished by their maximal regularity and almost crystalline symmetry: to each face of S, (read subspace), corresponds a unique ortho-complement, etc. Might this resemble the relation between the Euclidean and Riemannian geometries? If so then could it happen that in some circumstances the quantum systems could obey the generalized convex geometry, dissenting from the Hilbertian structure?

In the intents of finding a synthesis of the lattice ("logical") and probabilistic interpretations (since J. von Neumann [2]) the statistical aspects, in general, were subordinated to the assumed structure of the orthocomplemented lattice, and the answer of the axiomatic approach was always the same: the quantum mechanics must be exactly as it is. This belief turned even stronger due to the theorem of Gleason [10], as well as due to the profound and elegant generalizations of the algebraic approaches of Gel'fand and Naimark [11], Haag and Kastler [12], Pool [13], Araki [14], Haag [15], and other authors, who never resigned from the Hilbert space representations. Curiously, until today, these convictions find also a strong support in the well-known book of G. Mackey [16] in which, however, the axiomatic approach has some self-annihilating aspects: after a laborious presentation of six axioms on quantum logic $\mathcal{L}$, the seventh axiom tells flatly that the elements of the logic are closed vector subspaces of a Hilbert space, thus making all previous axioms redundant! (a short report on this school of axiomatics, see H. Primas [17, p. 211]). Some opposition is not so surprising…

The first descriptions of QM based exclusively on the convex geometry belong to G. Ludwig [18], though he adopted axioms in fact limiting the story to the orthodox scheme. The hypothesis about the possibility of quantum mixtures obeying non-Hilbertian geometries was formulated by the present author [6, 8], then by Davies and Lewis [19]. The hypothetical geometries succeeded to awake both positive and hostile reactions. Roger Penrose at some moment hoped that the atypical structures might tell something about the nonlinear graviton [20], though later on he complained [21] that they give a pure statistical interpretation, without any analytical entity behind (though inversely, the nonlinear graviton of Penrose is a pure analytical entity without any statistical interpretation!). T.W. Kibble and S. Randjbar-Daemi followed [8] describing the *classical gravity* in interaction with the generalized quantum structure [22]. Some other authors in philosophy of physics stay firmly on the ground of the orthodox theory. Nonetheless, they don't escape objections. While Putnam considers the orthocomplemented structure of Hilbert spaces the "truth of quantum mechanics" [23] (taking the side of Mackey?), John Bell and Bill Hallet [24] adopt the generalized design proposed in [6] to show the weakness of Putnam's argument. However, the deformed geometries, if real, must occur in some concrete physical circumstances. Where should we look for them?

As it seems, the most natural possibility is to look for nonlinear variants of quantum mechanics. In fact, already some simple nonlinear cases of the Schrödinger's equation admit non quadratic, positive, absolutely conservative quantities which could be used to define the probability densities [8]. The quantum mechanics with logarithmic non-linearity permits to define consistently the reduction of the wave packets [25]. Yet, as shown by Haag and Bannier [26], subsequently also in [27], the nonlinear wave equations lead to high mobility of quantum states, breaking the quantum impossibility principles.

The most basic difficulty was noticed by N. Gisin, who had shown that if the linear evolution law of quantum states were amended by adding some nonlinear operations, then the breaking of the mixture indistinguishability would make possible to read the instantaneous messages between parts of the entangled particle systems [28, 29]. The simplest case would occur in a variant of EPR experiment for the sequences of photon pairs in the singlet polarization state $|\Xi\rangle = \frac{1}{\sqrt{2}}(|\updownarrow\rangle|\leftrightarrow\rangle - |\leftrightarrow\rangle|\updownarrow\rangle)$ emitted in two opposite directions. According to the present day theory the polarization measurements on the left photons can produce at distance (due to the correlation mechanism) any desired mixture (3) of the right photons (or vice versa). As long as mixtures (3) are indistinguishable, this does not transmit information. However, if the observer of the right photon states could cause their nonlinear evolution, he could distinguish the quantum mixtures (3), thus reading hidden information and reconstructing without delay the measurements performed by his distant counterpart on the left EPR photons. So, is the nonlinear QM impossible?

Perhaps, we should not overestimate the axiomatic approaches. What they usually tell is that we cannot modify just one element of the theory, while leaving the whole rest intact. If in the last decade of XIX century some excellent axiomaticians tried to formulate reasonable axioms defining the space-time structure, they would prove beyond any doubt that the space-time must be Galilean! Yet, it is not. The deviations (in our normal conditions) are very small, but rather important…

What can be impossible in QM, is to conserve the orthodox representation of pure states as the "rays" in a complex Hilbert space, together with the tensor product formalism, and with the unitary background evolution, but to extend it by adding some nonlinear evolution operations and to expect that the instantaneous information transfers will be still blocked. However, the whole deduction might be already overloaded by too many axioms. If the evolution were extended by some nonlinear operations, then in the first place, we would loose the Hilbert space orthogonality together with the trace rules for probabilities even without worrying about the superluminal messages…

Returning to the spin or polarization qubits, the possibilities of generalizing the Hilbertian structures depends not so much on axioms but rather on precise knowledge of probabilities. If indeed exactly orthodox, then may be, the qubits can only rigidly rotate…

The problems of systems traditionally described by multi- or infinitely-dimensional Hilbert spaces are more difficult. The questions of Hans Primas, perhaps are still waiting for a good answer: *Does quantum mechanics apply to large molecular systems?… Why do so many stationary states not exist?* (see [18, pp. 11 and 12]). Indeed, even the problem of how to create in practice the one particle states described by arbitrary wave packets deserves systematic studies [30, 31, 32, 33].

As recently noticed, the non-linear modifications of quantum dynamics instead of just extending the techniques of the state manipulation might introduce *constrains*, with the restricted S no longer obeying the Hilbertian geometry [34]; an option which might be worth exploring.

All the attempts to see more freedom in quantum structures need some empirical criteria, which would permit to detect the new geometries if they exist. In case of classical state structures such criteria were found by John Bell, in form of Bell inequalities expressing the Boolean geometry of the state mixtures. Their breaking was the sign that the ensembles are non-classical. The problems of quantum ensembles, e.g., whether they indeed obey the Hilbert space geometry, are significantly more involved. The initiative of our colleagues [9] to describe them in terms of "apophatic" (forbidden) properties continues indeed the effort of John Bell on the new theory level. Some interesting cases might be the "cross sections" of S, resembling the "constrained QM" discussed in [34], and the projections (the collapsed S caused by deficiency of observables?). Simultaneously, the mathematical research presented in [7, 9] is an unexpected school of modesty for all of us, who believed to understand so well the property of nice objects called the "density matrices". Now it turns out that we did not even know the properties of the simple qutrit! Needless to say, should any of the "forbidden properties" be detected for any statistical ensemble in some physical conditions, this will be the proof that the theory is at the new conceptual level. Interesting, what about all that will think the physicists of XXII century?

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A Time of Arrival Operator on the Circle (Variations on Two Ideas)

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Dedicated to Bogdan Mielnik

Abstract. Using the orthodox Weyl-Wigner-Stratonovich-Cohen (WWSC) quantization rule we construct a time of arrival operator for a free particle on the circle. It is shown that this operator is self-adjoint but the careful analysis of its properties suggests that it cannot represent a 'physical' time of arrival observable. The problem of a time of arrival observable for the 'waiting screen' is also considered. A method of avoiding the quantum Zeno effect is proposed and the positive operator-valued measure (POV-measure) or the generalized positive operator-valued measure (GPOV-measure) describing quantum time of arrival observable for the waiting screen are found.

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1. Introduction

We begin with some sentences by St. Augustine taken from his 'Confessions' [1]:

> What, then, is time? If no one ask of me, I know; if I wish to explain to him who asks, I know not. (*Book* 11*, chapter XIV*)

and

> When, therefore, they say that things future are seen, it is not themselves, which as yet are not (that is which are future); but their causes or their signs perhaps are seen, the which already are. Therefore, to those already beholding them, they are not future, but present, from which future things conceived in the mind are foretold. (*Book* 11*, chapter XVIII*)

A shadow of these two phrases can be easily recognized in two contemporary works. One of them, '"Time operator": the challenge persists' by Bogdan Mielnik and Gabino Torres Vega [2] shows essential difficulties with a definition of a quantum time observable and the authors conclude that: 'While the future of the subject is unknown, it becomes clear, that all intents to obtain the *time observable* in the orthodox form of a self-adjoint operator (in spite of the best stratagems to avoid the Pauli theorem [...]) lead to a blind alley. The resulting operators are typically plagued by some little but persistent difficulties which might look accidental; besides they all suffer some basic defect which seems common for the whole family.' ([2], p. 90)

The main question of the second work 'The screen problem' by Bogdan Mielnik [3] can be stated as follows: 'One of the crucial statements of quantum mechanics is that the state vector contains complete non contradictory information about the system' [3, p. 1128], so Mielnik asks, where is the information about the time coordinate of the event of absorption of a wave packet by the waiting screen (see [3, Fig. 1]).

The problem of understanding time or, in particular, time of arrival as a quantum observable, and not as a parameter only, has a long history and a vast bibliography which starts with distinguished works by W. Pauli [4], Y. Aharonov and D. Bohm [5], M. Razavy [6], G.R. Allcock [7], E.P. Wigner [8], J. Kijowski [9], to mention some of them (see also a nice review of this matter by J.G. Muga and C.R. Leavens [10]).

Although a big effort has been done to solve the problem, we are still far from a convincing solution. We have no satisfactory time of arrival operator as it is very clearly stressed in Ref. [2] and we have no explicit solution of the waiting screen problem described in Ref. [3].

The aim of the present work is to study these two questions once more. In Section 2, using the 'orthodox' Weyl-Wigner-Stratonovich-Cohen (WWSC) quantization rule we find a time of arrival operator for a free particle on a circle. It is shown that this operator has nice mathematical properties, namely it is bounded, self-adjoint and of Hilbert-Schmidt type. However, it cannot be interpreted as the operator representing the physical time of arrival observable since it is 'plagued by some little but persistent difficulties.' In Section 3 we consider the waiting screen (detector) problem for a free particle. Using the 'orthodox' reduction of state assumption in quantum mechanics and avoiding the quantum Zeno effect we find a formula for the average time of arrival, which in turn defines the (*generalized*) *positive operator-valued measure* ((G)POV-measure).

Our considerations are similar to the ones related to the decoherent histories approach to quantum mechanics developed by J.J. Halliwell and J.M. Yearsley [11, 12].

The present paper has, in fact, the form of two variations on the themes given by Mielnik [3], then Mielnik and Torres Vega [2] and it is an honor and a great pleasure to dedicate these variations to Bogdan Mielnik on the occasion of his 75th birthday.

2. A time of arrival operator on the circle

Consider first a free particle on the x-axis. If the coordinate of the particle at the initial moment $t_0 = 0$ is x then the time of arrival of this particle at the point $X = 0$ (screen) in classical mechanics reads

$$T = -m\frac{x}{p}, \tag{1}$$

where m is the mass of the particle and p stands for its momentum. Quantization of (1) in the symmetric ordering leads to the *Aharonov-Bohm time of arrival operator* [5]

$$\hat{T} = -\frac{m}{2}\left(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x}\right) = -i\frac{m}{\hbar}\frac{1}{\sqrt{k}}\frac{d}{dk}\frac{1}{\sqrt{k}}, \quad k = \frac{p}{\hbar} \tag{2}$$

which is maximally symmetric but has no self-adjoint extensions. The natural way out from that difficulty has been found by N. Grot, C. Rovelli and R.S. Tate [13], and it consists in an appropriate regularization of the operator (2) in a small neighborhood of the singular point $k = 0$. Thus one gets the *regulated time of arrival operator*

$$\hat{T}_\varepsilon = -i\frac{m}{\hbar}\sqrt{f_\varepsilon(k)}\frac{d}{dk}\sqrt{f_\varepsilon(k)}, \tag{3}$$

where $\varepsilon > 0$ is an arbitrary small positive number and $f_\varepsilon(k)$ is a real bounded continuous function such that

$$f_\varepsilon(-k) = -f_\varepsilon(k) \;\;,\;\; f_\varepsilon(k) = \frac{1}{k} \text{ for } |k| > \varepsilon \;,\; \forall_{k\neq 0}\; f_\varepsilon(k) \neq 0 \tag{4}$$

(for instance $f_\varepsilon(k) = \frac{1}{k}$ for $|k| > \varepsilon$ and $f_\varepsilon(k) = \frac{k}{\varepsilon^2}$ for $|k| < \varepsilon$). It has been shown in [13] that $\hat{T}_\varepsilon$ is self-adjoint. This is a very good news. However, there are also bad news:

(i) J. Oppenheim, B. Reznik and W.G. Unruh [14] have shown that if the particle is in an eigenstate of $\hat{T}_\varepsilon$ corresponding to some eigenvalue τ of $\hat{T}_\varepsilon$, then at the moment τ, i.e., at the predict time of arrival at the screen this particle can be detected far away from the screen with probability $\frac{1}{2}$

(ii) Eigenstates $|\tau, \pm\rangle$ (note the degeneration !) of $\hat{T}_\varepsilon$ are not *time translation invariant*, i.e.,

$$\exp\left\{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right\}|\tau, \pm\rangle \neq |\tau - t, \pm\rangle. \tag{5}$$

(This is a consequence of Pauli's theorem [4].)

(iii) Eigenvalues τ of $\hat{T}_\varepsilon$ can be both positive and negative. It seems that from the experimental point of view the negative time of arrival, $\tau < 0$, is questionable in quantum mechanics.

The above-mentioned points show that one can hardly consider $\hat{T}_\varepsilon$ as a correct time of arrival operator. Our first idea is to avoid the objection iii.

To this end we propose to deal with a free particle on the circle. Let $-\pi < \Theta \leq \pi$ denote the angle coordinate of a particle at the moment $t = 0$ on the circle of radius r and let L be the angular momentum of the particle. Then the time of arrival of this particle at the point $\Theta = 0$ (screen) is given by the following function

$$T(\Theta, L) = mr^2 \cdot \begin{cases} -\frac{2\pi+\Theta}{L} & \text{for } \Theta < 0\,,\ L < 0 \\ -\frac{\Theta}{L} & \text{for } \Theta > 0\,,\ L < 0 \ \text{ or } \ \Theta < 0\,,\ L > 0 \\ \frac{2\pi-\Theta}{L} & \text{for } \Theta > 0\,,\ L > 0 \\ g(\Theta) \geq 0 & \text{for } L = 0. \end{cases} \tag{6}$$

Of course $T(\Theta, L)$ describes the *first passage time* [10].

An arbitrary non negative function $g(\Theta) \geq 0$ plays the analogous role as the function $f_\varepsilon(k)$ in (3), i.e., $g(\Theta)$ regularizes the classical function $T(\Theta, L)$ at the point $L = 0$. We quantize $T(\Theta, L)$ according to the WWSC method [15], [16], [17], [18]. Thus we arrive at the operator

$$\hat{T}_{(\mathbf{K})} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} T(\Theta, n\hbar)\, \hat{\Omega}_{(\mathbf{K})}(\Theta, n)\, \frac{d\Theta}{2\pi} \tag{7}$$

where $\hat{\Omega}_{(\mathbf{K})}(\Theta, n)$ is the *generalized Stratonovich-Weyl quantizer,* which in the case of a circle reads [19]–[23]

$$\hat{\Omega}_{(\mathbf{K})}(\Theta, n) = \sum_{l=-\infty}^{\infty} \int_{-\pi}^{\pi} \mathbf{K}(\sigma, l) \exp\{-i(\sigma n + l\Theta)\}\, \hat{U}(\sigma, l) \frac{d\sigma}{2\pi} \tag{8}$$

with

$$\begin{aligned} \hat{U}(\sigma, l) &= \exp\left\{-\frac{i}{2} l\sigma\right\} \exp\left\{\frac{i}{\hbar}\sigma\hat{L}\right\} \exp\left\{il\hat{\Theta}\right\} \\ &= \exp\left\{\frac{i}{2} l\sigma\right\} \exp\left\{il\hat{\Theta}\right\} \exp\left\{\frac{i}{\hbar}\sigma\hat{L}\right\} \\ &= \sum_{k=-\infty}^{\infty} \exp\left\{i\left(k + \frac{l}{2}\right)\sigma\right\} |k+l\rangle\langle k| \end{aligned} \tag{9}$$

where $|k\rangle, k = 0, \pm 1, \ldots,$ stands for the normalized eigenvector of $\hat{L}$

$$\hat{L}|k\rangle = k\hbar|k\rangle\ ,\ \langle k|k'\rangle = \delta_{kk'}. \tag{10}$$

The kernel function $\mathbf{K} = \mathbf{K}(\sigma, l),\ -\pi < \sigma \leq \pi,\ l \in \mathbb{Z}$ determines an ordering of operators. For example if $\mathbf{K} = 1$ then one gets the *Weyl ordering*, for $\mathbf{K} = \cos\left(\frac{l\sigma}{2}\right)$ one obtains the *symmetric ordering.* Therefore, using (7), (8) and (9) with $\mathbf{K} = \cos\left(\frac{l\sigma}{2}\right)$ and performing simple but rather tedious manipulations we find the time of arrival operator in the symmetric ordering for a free particle on the circle

$$\begin{aligned} \hat{T}_S = mr^2 \cdot \Bigg\{ & \frac{1}{2i\hbar} \sum_{\substack{j,k=-\infty,\\ j\neq 0,\, k\neq 0,\\ j\neq k}}^{\infty} \frac{j+k}{jk(j-k)} |j\rangle\langle k| + \frac{\pi}{\hbar} \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{1}{|k|} |k\rangle\langle k| \\ & + \sum_{\substack{k=-\infty,\\ k\neq 0}}^{\infty} \left[\frac{1}{2i\hbar k^2} + \frac{1}{4\pi} \int_{-\pi}^{\pi} g(\Theta) \exp\{-ik\Theta\} d\Theta \right] |k\rangle\langle 0| \\ & + \sum_{\substack{k=-\infty,\\ k\neq 0}}^{\infty} \left[-\frac{1}{2i\hbar k^2} + \frac{1}{4\pi} \int_{-\pi}^{\pi} g(\Theta) \exp\{ik\Theta\} d\Theta \right] |0\rangle\langle k| \\ & + \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\Theta) d\Theta |0\rangle\langle 0| \Bigg\}. \end{aligned} \tag{11}$$

One can show that the operator $\hat{T}_S$ has nice mathematical properties. It is defined on the all Hilbert space $L^2(S^1)$. Then it is self-adjoint, bounded and of Hilbert-Schmidt type so it is also a completely continuous (compact) operator. Hence, due to the Hilbert-Schmidt theorem $\hat{T}_S$ can be represented as follows

$$\hat{T}_S = \sum_{k=1}^{\infty} \tau_k |\tau_k\rangle\langle\tau_k|, \tag{12}$$

$$\tau_k \in \mathbb{R}\ ,\ \sum_{k=1}^{\infty} \tau_k^2 < \infty\ ,\ \langle\tau_k|\tau_l\rangle = \delta_{kl}\ ,\ \sum_{k=1}^{\infty} |\tau_k\rangle\langle\tau_k| = \hat{1}.$$

One can also show that the *time of arrival operator in the Weyl ordering* has the same properties. We expect that these properties will be recovered for any time of arrival operator of a free particle on the circle which is constructed by quantizing some classical time of arrival function corresponding to the *first passage time.*

Further analysis of the properties of the time of arrival operator $\hat{T}_S$ leads to the conclusions

(a) $\hat{T}_S$ has a discrete spectrum with the accumulation point 0. For every $\lambda > 0$ there exists a finite number of eigenvalues τ_k of $\hat{T}_S$ such that $|\tau_k| > \lambda$. The spectrum of $\hat{T}_S$ depends on the mass of the particle, what means, for instance, that the participants of the Białowieża conference are not able to arrive at Białowieża at the same time. Moreover, we should consider the 'clock time' which appears to be continuous and the arrival time which for a given particle is discreet.

(b) In general

$$\exp\left\{-\frac{i}{\hbar} t\hat{H}\right\} |\tau_k\rangle \not\sim |\tau_k - t\rangle \tag{13}$$

for any Hamiltonian $\hat{H}$, i.e., $\hat{T}_S$ is not a *time translation invariant* (compare with (5)).

(c) Numerical (computer) results show that for $g(\Theta) = \text{const.} \geq 0$ even so the classical time of arrival function $T(\Theta, p) \geq 0$ the operator $\hat{T}_S$ has positive as well as negative eigenvalues. The remedy for this could be the definition of the time of arrival operator $\sqrt{\hat{T}_S}$ (see also [11], [12]). However, this does not cure the lack of the time translation invariance. Moreover, our preliminary calculations lead to the arguments analogous to those by J. Oppenheim, B. Reznik and W.G. Unruh [14], i.e., assuming $g(\Theta) = \text{const.} \geq 0$, at the predict time of arrival the particle can be detected far away from the point $\Theta = 0$ (screen) with considerable probability.

Most likely the statements a–c hold true for any time of arrival operator constructed by the WWSC method from a classical time of arrival function for a free particle on the circle. Although further investigations for an arbitrary $g(\Theta)$ are needed (we are working on this problem) one can repeat Mielnik's and Torres Vega's words: '"Time operator" The challenge persists'.

Moreover, contrary to some suggestions [13], [24] it seems that one cannot 'forget time' and that x and t cannot be treated on equal footing in quantum mechanics.

3. A waiting screen

Here we deal with a particle in $\mathbb{R}^3$ which can be detected by a *waiting screen* (*detector*). We assume that the particle is absorbed (detected) if and only if it falls into some domain $V \subset \mathbb{R}^3$. Define two projectors

$$\hat{E} := \int_V |\vec{x}\rangle d^3x \langle \vec{x}| \ , \ \hat{E}' = \hat{1} - \hat{E}. \tag{14}$$

Consider then a time interval $[0, t]$ and choose the moments of time $0 = t_0 < t_1 < \cdots < t_n = t$. If the initial state of a particle is $|\Phi_{in}\rangle$, $\langle\Phi_{in}|\Phi_{in}\rangle = 1$, and one assumes the orthodox doctrine of quantum mechanics about a state reduction also for measurements performed without touching the object, then straightforward calculations show that the probability $\mathbf{P}_j$, $j = 0, 1, \ldots, n$ of absorption at the moment t_j reads

$$\begin{aligned} \mathbf{P}_j = \langle\Phi_{in}|\hat{E}' \exp\left\{\frac{i}{\hbar}(t_1 - t_0)\hat{H}\right\} \hat{E}' \exp\left\{\frac{i}{\hbar}(t_2 - t_1)\hat{H}\right\} \\ \cdots \hat{E}' \exp\left\{\frac{i}{\hbar}(t_j - t_{j-1})\hat{H}\right\} \hat{E} \exp\left\{-\frac{i}{\hbar}(t_j - t_{j-1})\hat{H}\right\} \hat{E}' \\ \cdots \exp\left\{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}\right\} \hat{E}' \exp\left\{-\frac{i}{\hbar}(t_1 - t_0)\hat{H}\right\} \hat{E}'|\Phi_{in}\rangle, \end{aligned} \tag{15}$$

where $\hat{H}$ is the Hamiltonian (see B. Misra and E.C.G. Sudarshan [25] and [11, 12]).

Taking $t_j - t_{j-1} = \frac{t}{n}$, $j = 1, \ldots, n$ we obtain

$$\mathbf{P}_j = \langle \Phi_{in} | \left(\hat{E}' \exp\left\{ \frac{i}{\hbar} \frac{t}{n} \hat{H} \right\} \right)^j \hat{E} \left(\exp\left\{ -\frac{i}{\hbar} \frac{t}{n} \hat{H} \right\} \hat{E}' \right)^j | \Phi_{in} \rangle. \tag{16}$$

If $\hat{H}$ is self-adjoint and semi-bounded then [25], [26]

$$\lim_{n \to \infty} \mathbf{P}_n = 0. \tag{17}$$

This is, of course, the famous *quantum Zeno effect* which in our case states that if the particle is not absorbed at the moment $t_0 = 0$ then it will not be absorbed at all. To avoid this paradoxical statement one can assume that

(I) $\hat{E}'$ is not a projector $\hat{1} - \hat{E}$. This corresponds to the assumption that there exists a complex potential [7], [11], $-iV_0$, $V_0 > 0$, such that

$$\hat{E}' = \exp\left\{ -\frac{i}{\hbar} \frac{t}{n} (-iV_0 \hat{E}) \right\} = \hat{1} - \hat{E} + \exp\left\{ -\frac{V_0}{\hbar} \frac{t}{n} \right\} \hat{E} \ , \quad \frac{V_0}{\hbar} \frac{t}{n} \gg 1. \tag{18}$$

Alternatively one can assume that only a *partial state reduction* has place when the measurement without interaction is performed and $\hat{E}'$ describes such a partial state reduction ($\hat{E}' \cdot \hat{E}' \neq \hat{E}'$).

(II) Continuous measurement is not allowed and

$$\eta := \frac{t}{n} > \tau_z = \left(\langle \Phi_{in} | \hat{H}^2 | \Phi_{in} \rangle - (\langle \Phi_{in} | \hat{H} | \Phi_{in} \rangle)^2 \right)^{-\frac{1}{2}} \hbar \tag{19}$$

where τ_z is the *Zeno time*.

Note that Mielnik considers in [3] this last argument as 'visibly unfair' (see [3, p. 1123]). We guess that it is not so unfair if one takes into account that any measurement device has a specific dead time. The assumption (19) is given also in [11, 12].

Suppose that

$$\sum_{j=0}^{\infty} \mathbf{P}_j = 1 \tag{20}$$

where from (16) with (19) we have

$$\mathbf{P}_j = \langle \Phi_{in} | \left(\hat{E}' \exp\left\{ \frac{i}{\hbar} \eta \hat{H} \right\} \right)^j \hat{E} \left(\exp\left\{ -\frac{i}{\hbar} \eta \hat{H} \right\} \hat{E}' \right)^j | \Phi_{in} \rangle. \tag{21}$$

Then the *average time of arrival* reads

$$\langle \tau \rangle = \sum_{j=0}^{\infty} j \eta \mathbf{P}_j. \tag{22}$$

Therefore, one arrives at the conclusion that in the present case quantum time of arrival is defined by the *positive operator-valued measure* (POV-measure)

$$\mathbb{N} \ni j \longmapsto \left(\hat{E}' \exp\left\{ \frac{i}{\hbar} \eta \hat{H} \right\} \right)^j \hat{E} \left(\exp\left\{ -\frac{i}{\hbar} \eta \hat{H} \right\} \hat{E}' \right)^j \quad \mathbb{N} = \{0, 1, \ldots\}. \tag{23}$$

(About POV-measures see, e.g., [27, 28].) If (20) does not hold, i.e., the particle can be not absorbed at all then $\sum_{j=0}^{\infty} \mathbf{P}_j < 1$ and the average time of arrival can be defined as

$$\langle \tau \rangle = \frac{\sum_{j=0}^{\infty} j\eta \mathbf{P}_j}{\sum_{j=0}^{\infty} \mathbf{P}_j} \tag{24}$$

and, consequently, the formula (23) gives now a *generalized positive operator-valued measure* (GPOV-measure).

All results given hitherto in this section can be easily generalized on the case of a particle moving on some submanifold of $\mathbb{R}^3$. In particular one can quickly carry over the last result to the case of a particle on the circle with the waiting screen. Assuming now that we deal with multiple crossing the screen by the particle we can state that (20) holds true and, consequently, quantum time of arrival for a particle on the circle is given by the POV-measure (23).

Finally, according to our considerations, a partial answer to the question asked by Mielnik in [3] could be the following: *The information about the time coordinate of the event of absorption of a wave packet by the waiting screen is contained in the formula* (22) *or, in general, in* (24).

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Negative Time Delay for Wave Reflection from a One-dimensional Semi-harmonic Well

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To Professor Bogdan Mielnik with our deepest admiration.

Abstract. It is reported that the phase time of particles which are reflected by a one-dimensional semi-harmonic well includes a time delay term which is negative for definite intervals of the incoming energy. In this interval, the absolute value of the negative time delay becomes larger as the incident energy becomes smaller. The model is a rectangular well with zero potential energy at its right and a harmonic-like interaction at its left.

Mathematics Subject Classification (2010). 35Q40; 35B34; 81U30; 81Q60.

Keywords. Exactly solvable potentials, phase time.

The time taken by a particle to traverse a given spatial region is one of the most striking features of quantum theory [1, 2]. In the case of tunneling through a one-dimensional barrier of height V_0 and width ξ, the *transmission time* of a wave packet centered at the average total energy $E = \hbar\omega = \hbar^2 k^2/(2m) < V_0$ is independent of the barrier thickness [3]. Thus, the peak value of the packet propagates with the effective group velocity $v_g = d\omega/dk = \hbar k/m$, which must increase with ξ across the barrier. Using electromagnetic analogues, superluminal ("anomalously large") group velocities have been observed for evanescent modes [4], microwave pulses [5], and in the tunneling of photons through 1D photonic band gaps [6]. Indeed, this 'abnormal behavior' of light [7] has stimulated the designing of high-speed devices based on the tunneling properties of semiconductors (see, e.g., Chs. 11 and 12 of Ref. [1]). In the stationary phase approximation [8], the *phase time* (group delay) is defined as the energy derivative of the transmission phase $\tau_W = \hbar \frac{d\varphi}{dE} = \frac{1}{v_g}\frac{d\varphi}{dk}$. This gives information of the time taken by the peak of the transmitted packet to appear, measured from the moment the peak of the incident packet strikes a given barrier. Another well-established notion of time considers the average time spent by the particles in the barrier. It is called the *dwell time* and is defined as the ratio $\tau_D = n/j$, with n the number of particles within the barrier and j the incident flux [9]. Yet, τ_W and τ_D are not necessarily related with each other; they are comparable only if the barrier is almost transparent [10].

While the quantum tunneling of rectangular barriers has attracted a lot of attention in recent years (see, e.g., [11,12] and references quoted therein), the scattering properties of rectangular wells have been underestimated. Quite recently, however, nonevanescent propagation has been predicted for potential wells [13]. In contradistinction with the tunneling exponential attenuation, the scattering at quantum wells attenuates the outgoing wave packets only because of the multiple reflections at the well boundaries. Negative phase times are then expected under certain conditions of the incident energy and the thickness of the well [13,14], a phenomenon which should be observable for electromagnetic wave propagation [15]. Thereby, rectangular wells may lead to much larger advancements than rectangular barriers in the context of traversal times [16].

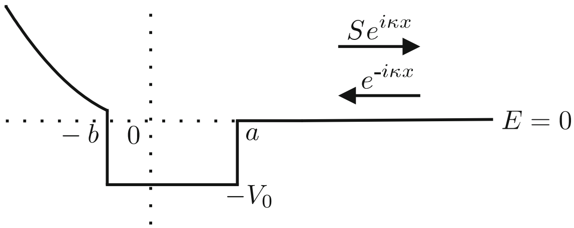

Figure 1. Schematic representation of the one-dimensional semi-harmonic square well as a function of the dimensionless position x. The wave e^{-ikx} colliding the well from the right is reflected to give Se^{ikx}, with $S = e^{i\delta}$ the reflection amplitude and $\delta(E)$ the reflection phase shift.

The purpose of this contribution is to report negative time delay for a one-dimensional well which reduces the scattering process to the case of purely wave reflection. The absolute value of this negative time delay becomes larger as the energy of the incident particle becomes smaller. To begin with, consider the stationary Schrödinger equation $(H-E)\psi(x) = 0$, where $V(x)$ is the one-dimensional potential depicted in Figure 1. This last is a rectangular well in a semi-harmonic background integrated by zero potential energy (flat potential) to the right and a harmonic-like potential to the left of the well. Our model corresponds to a system (the rectangular well) embedded in an environment (the parabolic plus flat potentials), and the issue is the study of the modifications on the physical properties of the system due to the environment [17]. For instance, the number $N+1$ of bound states $\psi_n(x)$, $n = 0, 1, \ldots, N$, is determined by the area $A = (a+b)V_0$ of the rectangular well. Here, $a+b$ and $-V_0$ are respectively the width and depth of the well with $V_0 > 0, a \geq 0$, and $b \geq 0$. Once the semi-harmonic background is added, the number $N+1$ is preserved but the corresponding energies $E_0, E_1, \ldots, E_N$, are displaced towards the positive threshold. This last property does not depend on the geometry of the rectangle; the wells having the same area admit the same number of bound states. In this context, remark that the wells of unit area $V_0 = a+b$ admit only one bound state and constitute a family of compact support functions which converge to the delta well in the sense of distribution theory [18]. Then, the single bound state (dimensionless) energy $E_0 = -0.25$ of the delta well becomes less negative $E_0 = -0.0797104$ in the presence of the semi-harmonic background [17] (compare with [19]).

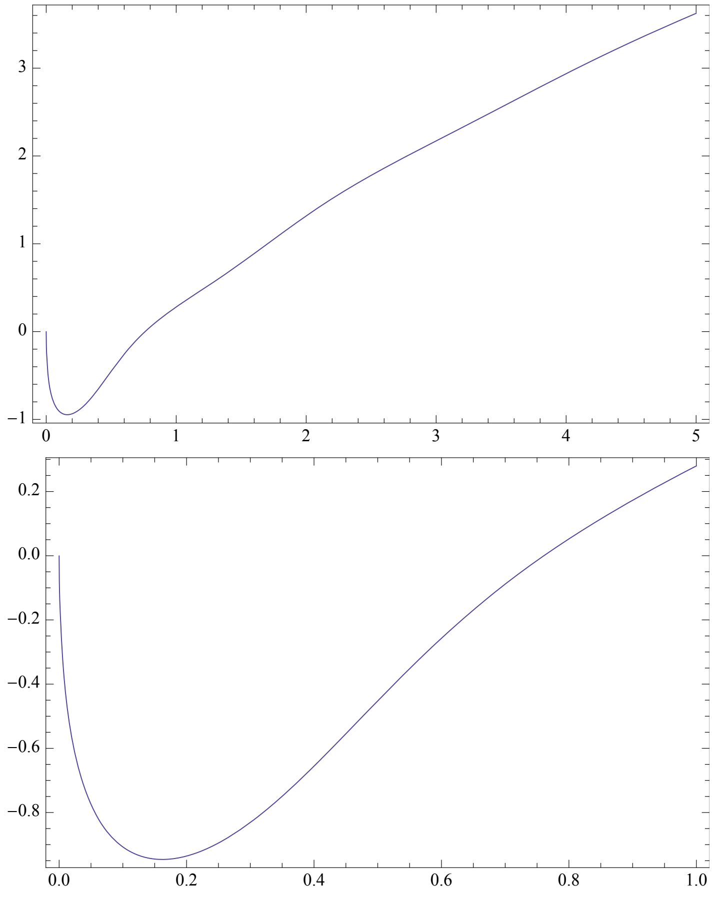

FIGURE 2. The reflection phase shift $\delta(E)$ of a semi-harmonic well of unit area as a function of the dimensionless energy E for the parameters $a = b = 5/2$, and $V_0 = 1/5$. A detail of the behavior of $\delta(E)$ for low energies is shown at the bottom figure.

On the other hand, the isolated resonances of a rectangular well are easily identified by expressing the transmission amplitude T as a superposition of Breit-Wigner distributions [20]. The center $E_k^r > 0$ and width Γ_k of each of these peaks

a	E_a	a	E_a
2.5	0.03406092	1.0	0.10100123
2.0	0.05056413	0.5	0.16473112
1.5	0.07205970	0.0	0.45727096

TABLE 1. The (dimensionless) energy E_a defining the change of sign in the time delay for a semi-harmonic well of unit area.

define the resonance complex eigenvalue $\epsilon_k = E_k^r - i\Gamma_k/2$, and induce time delays in the scattering process [21]. A rapid increasing of the transmission phase is then expected in the vicinity of the resonance position E_k^r. According to Wigner, the increases of the phase should be balanced by the appropriate decreases [8]. Therefore, the slope of the transmission phase can be even negative in order to compensate for the phase increases associated with each of the resonances. This effect is more important near the energy threshold, below the position of the first Breit-Wigner peak of T [14]. In other words, the negative phase times predicted in [13] are in complete agreement with the conditions to get at least one isolated resonance in rectangular wells [14, 20]. If the semi-harmonic environment is activated, all the scattering states become more excited and their wave functions cancel at $x = -\infty$. As the potential includes neither sources nor shrinks, the probability is conserved and all the incoming waves are reflected. Then, the reflection phase shift $\delta(E)$ encodes all the information of the scattering process. This phase is depicted in Figure 2 for a unit area semi-harmonic square well with $a = b = 5/2$. Notice the strong negative slope in the interval of dimensionless energies $(0, 0.16208517)$, so that negative time delay is expected for wave packets colliding the well from the right at the appropriate energy.

The straightforward calculation shows that the phase time is given by $\tau_W = \tau_p - \tau_E$, with $\tau_p = 2a/v_g$ the classical flight time to traverse a distance $2a$, and the time delay τ_E written in the form

$$\tau_E = \frac{1}{v_g}\frac{\partial}{\partial k}\left[\arctan\left(\frac{2\phi_1\phi_2}{\phi_1^2 - \phi_2^2}\right)\right].$$

Here the functions ϕ_1 and ϕ_2 are given by

$$\phi_1 = -\frac{q}{2}\sin 2qa + \left.\frac{\psi'}{\psi}\right|_{x=-a}\cos 2qa, \qquad \phi_2 = -k\cos 2qa - \frac{1}{q}\left.\frac{\psi'}{\psi}\right|_{x=-a}\sin 2qa,$$

with

$$\psi(x) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-k^2}{4}, \frac{1}{2}; x^2\right) + 2x\frac{\Gamma(\frac{3-k^2}{4})}{\Gamma(\frac{1-k^2}{4})}{}_1F_1\left(\frac{3-k^2}{4}, \frac{3}{2}; x^2\right)\right],$$

and $q = \sqrt{V_0 + k^2}$. The expression ${}_1F_1(a, c; z)$ stands for the confluent hypergeometric function.

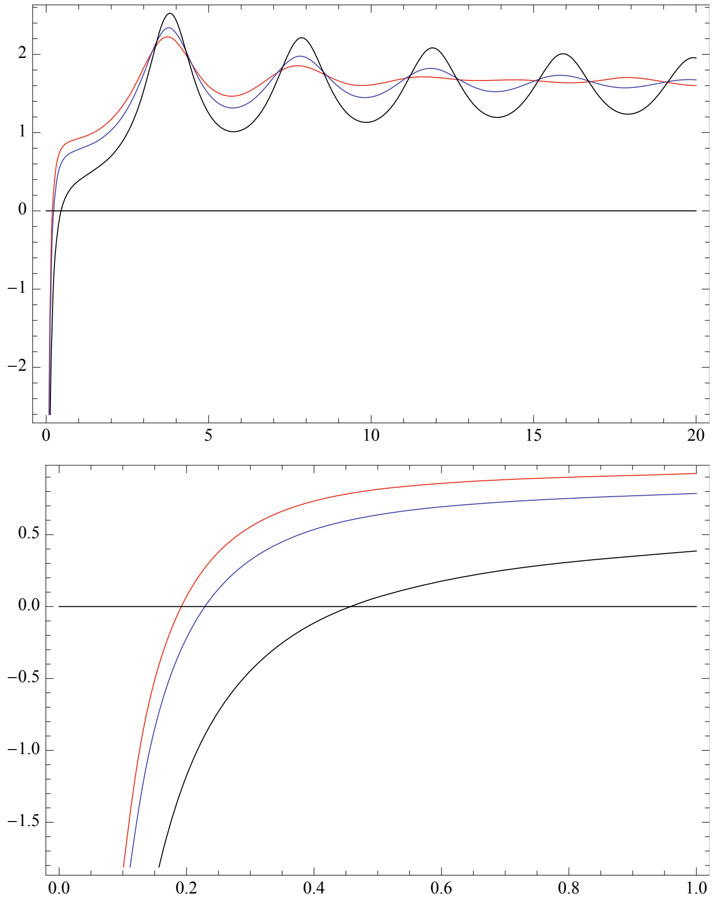

FIGURE 3. Time delay τ_E of a unit area semi-harmonic well for $a = 0.4$ (red curve), $a = 0.03$ (blue curve) and $a = 0$ (black curve). A detail of the behavior at low energies is shown at the bottom figure.

Figure 3 shows the behavior of the time delay τ_E for some semi-harmonic wells of unit area but different geometries. Given a, there is an interval of scattering energies $(0, E_a)$ where τ_E is negative (for definite values see Table 1). In this interval, the absolute value of the negative time delay becomes larger as the incident energy becomes smaller. Thus, it is clear the dependence of τ_E on the energy E of the incident particles and on the rectangular well thickness $2a$. For a given value of a, the maxima of the time delay are localized at the real part of the resonance eigenvalues $\epsilon_k = E_k^r - i\Gamma_k/2$, as expected. The energies E_k^r are displaced to more excited values as $a \to 0$. In the very limit $a = 0$, the time delay changes its sign at the scattering energy $E_{a=0} = 0.45727096$ and oscillates around the asymptotic value $\frac{\pi}{2}$ for $E > E_{a=0}$. It should be pointed out that the interval of scattering energies $(0, E_{a=0})$ is the largest one in which τ_E is negative for any of the unit area semi-harmonic wells (see Table 1 and Figure 3).

Let us close this contribution with some remarks on the optical analogs applied in the study of particles passing through a rectangular well [13,15]. Of particular interest, negative phase times have been confirmed for electromagnetic wave propagation in waveguides filled with different dielectrics [15]. The negative time delay τ_E of the semi-harmonic wells could be studied in a similar way by taking $b = 0$ and $a \geq 0$. Once the energy baseline of the rectangular well is shifted by the constant value $E_0 = \hbar\omega_0$, the cutoff frequency ω_0 of the first waveguide section is defined. Then, waveguide sections with different cutoff frequencies can be constructed to approximate the parabolic part of the potential by a series of Riemann rectangles. As a result, the semi-harmonic well can be connected to a piecewise frequency $\omega_c(x)$. Following [15], the solution to the propagation problem (i.e., the Helmholtz equation for ω_c) is obtained if the wave functions and the electromagnetic fields satisfy identical boundary conditions. Further details will be given elsewhere.

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Characterizing Non-Markovian Dynamics

D. Chruściński and A. Kossakowski

Dedicated to honor Professor Woronowicz on the occasion of his 70th birthday.

Abstract. We characterize (non)Markovian dynamics of open quantum systems. Two recently proposed measures of non-Markovianity are analyzed: one based on the concept of divisibility of the dynamical map and the other one based on distinguishability of quantum states. The characterization of the corresponding generators in the Heisenberg picture is provided as well.

Mathematics Subject Classification (2010). Primary 47L05; Secondary 81Q05.

Keywords. Operator algebras, quantum dynamics, Markovian evolution.

1. Introduction

The dynamics of open quantum systems attracts nowadays increasing attention [1–3]. It is relevant not only for better understanding of quantum theory but it is fundamental in various modern applications of quantum mechanics. Since the system-environment interaction causes dissipation, decay and decoherence it is clear that dynamics of open systems is fundamental in modern quantum technologies, such as quantum communication, cryptography and computation [4]. The usual approach to the dynamics of an open quantum system consists in applying an appropriate Born-Markov approximation leading to the celebrated quantum Markov semigroup [5, 6] which neglects all memory effects. However, recent theoretical studies and technological progress call for a more refined approach based on non-Markovian evolution.

Non-Markovian systems appear in many branches of physics, such as quantum optics [1, 7], solid state physics, quantum chemistry, and quantum information processing. Since non-Markovian dynamics modifies monotonic decay of quantum coherence it turns out that when applied to composite systems it may protect quantum entanglement for longer time than standard Markovian evolution. It is therefore not surprising that the non-Markovian dynamics was intensively studied during last years [8–19].

In the present paper we perform further analysis of this problem. We analyze two recently proposed measures of non-Markovianity: one based on the concept of divisibility of the dynamical map and the other one based on the distinguishability of quantum states. Let us observe that the evolution of the system living in the Hilbert space $\mathcal{H}$ may be considered as a reduced dynamics of some composed system living in $\mathcal{H} \otimes \mathcal{H}_{\rm R}$ governed by the Hamiltonian H. If ω is a fixed state of the reservoir then one may define

$$\rho_t = {\rm tr}_{\rm R}\, [e^{-iHt}(\rho \otimes \omega)e^{itH}] \ , \tag{1}$$

where ρ is an initial state of the system and ${\rm tr}_{\rm R}$ denotes the partial trace over the reservoir degrees of freedom. The above formula establishes a linear map $\Lambda_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\rho_t = \Lambda_t \rho$. This map may be regarded as a mathematical representation of the system evolution. We stress that Λ_t defined by (1) is completely positive and trace preserving (see Section 2) but it does not possesses any further special properties. In particular it is not true that Λ_t satisfies so-called composition law: $\Lambda_{t+u} = \Lambda_t \Lambda_u$ for $t, u \geq 0$. Only after suitable approximation the formula (1) may lead to a Markovian semigroup satisfying composition law.

In this paper we study further properties of the dynamical map Λ_t. In particular we analyze when Λ_t defines a (non)Markovian evolution.

2. Positive linear maps

Let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a linear map from the $\mathbb{C}^*$-algebra $\mathcal{A}$ into the space of bounded operators in the Hilbert space $\mathcal{H}$. A map Φ is hermitian iff $\Phi(a^*) = (\Phi(a))^*$. One calls Φ a positive map [20] if $\Phi(a) \geq 0$ for all $a \geq 0$. Any positive map is necessarily hermitian. A map Φ is k-positive if

$${\rm id}_k \otimes \Phi : M_k \otimes \mathcal{A} \to M_k \otimes \mathcal{B}(\mathcal{H}) \ , \tag{2}$$

is positive. In the above formula ${\rm id}_k$ denotes the identity map in the algebra of $k \times k$ complex matrices M_k. Finally, Φ is completely positive (CP) if it is positive for $k = 1, 2, 3, \ldots$. Due to the celebrated Stinespring theorem [20] CP maps are fully characterized: Φ is CP if there exists a Hilbert space $\mathcal{K}$ and a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that

$$\Phi(a) = V\pi(a)V^\dagger \ , \tag{3}$$

for some linear operator $V : \mathcal{K} \to \mathcal{H}$. If $\mathcal{A} = \mathcal{B}(\mathcal{H}')$ and both $\mathcal{H}$ and $\mathcal{H}'$ are finite-dimensional, then the Stinespring representation implies the existence of a set of so-called Kraus operators $K_\alpha : \mathcal{H}' \to \mathcal{H}$ such that

$$\Phi(a) = \sum_\alpha K_\alpha a K_\alpha^\dagger \ . \tag{4}$$

Note that Φ is trace preserving if $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}$ (by CPT we denote CP trace preserving maps). Moreover, Φ is unital, i.e., $\Phi(\mathbb{I}) = \mathbb{I}$, if $\sum_\alpha K_\alpha K_\alpha^\dagger = \mathbb{I}$. Interestingly, in spite of considerable effort, the structure of positive maps is rather poorly understood. Positive but not CP maps play an important role in entanglement theory [21]. Recall, that a state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if

$$\rho = \sum_\alpha p_\alpha \, \rho_\alpha^{(A)} \otimes \rho_\alpha^{(B)} \ , \tag{5}$$

where $p_\alpha \geq 0$ with $\sum_\alpha p_\alpha = 1$, and $\rho_\alpha^{(A)}$, $\rho_\alpha^{(B)}$ are density operators in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. One proves [21] the following

Proposition 1. *A state* $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *is separable iff* $(\mathrm{id} \otimes \Phi)\rho \geq 0$ *for all positive maps* $\Phi : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$.

Let $\{e_1, \ldots, e_n\}$ denote an orthonormal basis in $\mathcal{H}$ and introduce $|\psi^+\rangle = n^{-1/2} \sum_{i=1}^n e_i \otimes e_i$ together with the corresponding projector $P^+ = |\psi^+\rangle\langle\psi^+|$. Recall that $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is CP if the so-called Choi matrix

$$C_\Phi := (\mathrm{id} \otimes \Phi) P^+ \ , \tag{6}$$

is semi-positive definite. Now, if Φ is trace preserving then $\mathrm{tr}\, C_\Phi = 1$ and Φ is CPT iff $||C_\Phi||_1 = 1$, where $||a||_1 = \mathrm{tr}\,|a|$ denotes the trace norm. The simplest example of positive but not CP map is a matrix transposition $\tau : M_n \rightarrow M_n$ in a given basis: $\tau(A) = A^{\mathrm{R}}$. One finds for $n = 2$

$$C_\tau = (\mathrm{id} \otimes \tau) P^+ = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{7}$$

which is not positive definite and hence τ being a positive map is not CP. A positive map Φ is decomposable if

$$\Phi = \Phi_1 + \Phi_2 \circ \tau \ , \tag{8}$$

where Φ_1 and Φ_2 are CP. It was shown by Woronowicz [22] that all positive maps $\Phi : M_m \rightarrow M_n$ with (m, n) given by $(2, 2)$, $(2, 3)$ and $(3, 2)$ are decomposable. It is not known how to construct positive maps which are not decomposable (see recent papers [23–25]).

3. Dynamical maps

Consider now a quantum system living in a n-dimensional Hilbert space $\mathcal{H}$.

Definition 1. By the time evolution of a quantum system we mean a family of CPT maps $\Lambda_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ for $t \geq 0$ such that $\Lambda_0 = \mathrm{id}$.

The simplest example of a dynamical map consists in unitary evolution

$$\Lambda_t \rho := U_t \rho U_t^\dagger \ , \tag{9}$$

where $U_t = e^{-iHt}$. If ρ represents an initial state then $\rho_t := \Lambda_t \rho$ defines its time evolution and it satisfies the standard von-Neumann equation

$$i\dot{\rho}_t = [H, \rho_t] \ , \quad \rho_0 = \rho \ . \tag{10}$$

Actually, unitary dynamics is reversible $\Lambda_t^{-1} := \Lambda_{-t}$ and hence Λ_t is defined for all $t \in \mathbb{R}$. Much more sophisticated example is provided by the Markovian semigroup which generalizes unitary evolution. In this case $\Lambda_t := e^{Lt}$, where the generator L is defined by

$$L\,\rho = -i[H,\rho] + \frac{1}{2}\sum_\alpha \left([V_\alpha, \rho V_\alpha^\dagger] + [V_\alpha \rho, V_\alpha^\dagger]\right) \ . \tag{11}$$

In the above formula H represents system Hamiltonian and $V_\alpha : \mathcal{H} \rightarrow \mathcal{H}$ is the collection of arbitrary operators encoding the interaction between the system living in $\mathcal{H}$ and the environment. One proves [5,6] that $\Lambda_t = e^{Lt}$ is CPT if and only if L is defined by (11). We stress that Λ_t is no longer reversible, i.e., Λ_{-t} is no longer CP (unless all $V_\alpha = 0$).

Consider a general quantum evolution described by

$$\dot{\Lambda}_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathrm{id} \ . \tag{12}$$

One of the main problems in the theory of open systems dynamics is to characterize properties of time-dependent generator L_t which gives rise to a legitimate quantum dynamics, that is,

$$\Lambda_t = \mathrm{T}\ \exp\left(\int_0^t L_u du\right) \ , \tag{13}$$

is CPT, where T denotes chronological operator.

4. Markovianity versus divisibility

Definition 2. Dynamical map Λ_t is divisible if

$$\Lambda_s = V_{t,s}\, \Lambda_t \ , \tag{14}$$

where the propagators $V_{t,s}$ are CPT for all $t \geq s \geq 0$.

This mathematical property enables one to introduce the notion of Markovianity

Definition 3. Quantum evolution represented by the dynamical map Λ_t is Markovian if Λ_t is divisible.

It is clear that both unitary dynamics and Markovian semigroup satisfy (14) and hence they are Markovian. Moreover, $V_{s,t} = V_{s-t} := \mathrm{e}^{(s-t)L}$. Let us observe that any dynamical map Λ_t satisfies the following local equation

$$\dot{\Lambda}_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathrm{id} \ , \tag{15}$$

with some time-dependent generator L_t. Knowing Λ_t one formally finds the following formula for the corresponding generator $L_t = \dot{\Lambda}_t \Lambda_t^{-1}$, where we assume the existence of the inverse Λ_t^{-1}. Note, however, that even if Λ_t^{-1} exists it needs not be CPT. Now, if Λ_t is divisible one obtains the following equation for the propagator

$$\partial_t V_{t,s} = L_t V_{t,s} \ , \quad V_{s,s} = \mathrm{id} \ . \tag{16}$$

One has the following

Theorem 2. *$V_{t,s}$ is CPT iff L_t is of the Lindblad form for all $t \geq s$.*

It shows that Markovian dynamics is fully characterized by the properties of the corresponding local generator L_t.

$V_{t,s}$ is CPT iff $||(\mathrm{id} \otimes V_{t,s})P^+||_1 = 1$. It is shown [17] that the quantity

$$g(t) = \lim_{\epsilon \to 0+} \frac{||(\mathrm{id} \otimes V_{t+\epsilon,t})P^+||_1 - 1}{\epsilon} \ , \tag{17}$$

enjoys $g(t) > 0$ if and only if the original map Λ_t is indivisible. This formula may be equivalently rewritten in terms of the local generator L_t:

$$g(t) = \lim_{\epsilon \to 0+} \frac{||P^+ + \epsilon(\mathrm{id} \otimes L_t)P^+||_1 - 1}{\epsilon} \ . \tag{18}$$

Hence one may define the following measure [17]

$$\mathcal{N}(\Lambda_t) = \frac{\mathcal{I}}{\mathcal{I}+1} \ , \tag{19}$$

where $\mathcal{I} = \int_0^\infty g(t)dt$.

Example. Consider the following generator in $\mathcal{B}(\mathbb{C}^2)$:

$$L_t \rho = \frac{1}{2}\gamma(t)(\sigma_3 \rho \sigma_3 - \rho) \ , \tag{20}$$

where σ_k denote Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Note that $L_t \mathbb{I} = L_t \sigma_3 = 0$ and

$$L_t \sigma_1 = -\gamma(t)\sigma_1 \ , \quad L_t \sigma_2 = -\gamma(t)\sigma_2 \ , \tag{21}$$

One easily finds for the dynamics

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \longrightarrow \rho_t = \begin{pmatrix} \rho_{11} & e^{-\Gamma(t)}\rho_{12} \\ e^{-\Gamma(t)}\rho_{21} & \rho_{22} \end{pmatrix} \ , \tag{22}$$

where

$$\Gamma(t) := \int_0^t \gamma(u)du \ . \tag{23}$$

It is clear that L_t generates the Markovian semigroup if $\gamma(t) = \gamma_0 > 0$. It generates Markovian dynamics if $\gamma(t) \geq 0$ for all $t \geq 0$. Finally, L_t provides legitimate generator if $\Gamma(t) \geq 0$. Hence, if $\Gamma(t) \geq 0$ but $\gamma(t)$ attains strictly negative values the corresponding dynamics is truly non-Markovian. Taking for example $\gamma(t) = \gamma_0 \sin t$ one finds $\Gamma(t) = \gamma_0(1 - \cos t) \geq 0$. And hence the evolution is non-Markovian (even periodic, $\Gamma(t + 2\pi) = \Gamma(t)$).

This simple example shows that Markovian evolution defines only a special class of quantum evolution characterized by the special property of L_t. The generic evolution is non-Markovian and the corresponding properties of L_t are not known.

5. Markovianity versus flow of information

Recently, another criterion of non-Markovianity was proposed by Breuer, Laine and Piilo in [16]. This criterion identifies non-Markovian dynamics with certain physical features of the system-reservoir interaction. They define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system. This backflow of information may manifest itself as an increase in the distinguishability of pairs of evolving quantum states. Recall, that if $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear positive trace preserving map then

$$||\Phi\rho_1 - \Phi\rho_2||_1 \leq ||\rho_1 - \rho_2||_1 \ , \tag{24}$$

for any pair of density operators $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H})$. It shows that

$$||\Lambda_t(\rho_1 - \rho_2)||_1 \leq ||\rho_1 - \rho_2||_1 \ , \tag{25}$$

that is, the distinguishability of quantum states $D[\rho_1, \rho_2]$ measured by the trace distance

$$D[\rho_1, \rho_2] := ||\rho_1 - \rho_2||_1 \ , \tag{26}$$

never increases in time. BLP [16] define the flux of information

$$\sigma(\rho_1, \rho_2; t) := \frac{d}{dt} ||\Lambda_t(\rho_1 - \rho_2)||_1 \ , \tag{27}$$

to control the time evolution of $||\Lambda_t(\rho_1 - \rho_2)||_1$. It is easy to show that for unitary dynamics $\sigma(\rho_1, \rho_2; t) = 0$, whereas for the Markovian semigroup $\sigma(\rho_1, \rho_2; t) < 0$. It is therefore natural to adopt the following definition [16]: evolution Λ_t is Markovian iff $\sigma(\rho_1, \rho_2; t) \leq 0$ for all pairs of initial states ρ_1 and ρ_2, and all $t \geq 0$.

Now comes the natural question: how these two definitions of Markovianity are related. It turns out that if Λ_t is divisible then $\sigma(\rho_1, \rho_2; t) \leq 0$. However, the converse is not true. It was shown recently [19] that it is possible to construct a simple model of quantum dynamics of a 2-level system such that $\sigma(\rho_1, \rho_2; t) \leq 0$ but Λ_t is not divisible. However, both approaches to (non)Markovianity may be easily reconciled [19]: let us define

$$||\Phi||_1 := \sup_{||a||_1 = 1} ||\Phi(a)||_1 \ , \tag{28}$$

and so-called *diamond norm*

$$||\Phi||_\diamond := ||\,\mathrm{id} \otimes \Phi||_1 \ . \tag{29}$$

One proves

Theorem 3. *The following conditions are equivalent*

1. $V_{t,s}$ *is CPT, i.e.,* Λ_t *is divisible,*
2. $V_{t,s}$ *satisfies* $||(\mathrm{id} \otimes V_{t,s})X||_1 \leq ||X||_1$ *for* $X^\dagger = X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$*,*
3. $||V_{t,s}||_\diamond = 1$*.*

Note, that introducing *complete information flow*

$$\widetilde{\sigma}(X;t) := \frac{d}{dt}\,||(\mathrm{id}\otimes\Lambda_t)X||_1\ , \tag{30}$$

one has the following

Corollary 4. *Λ_t is divisible iff $\widetilde{\sigma}(X;t)\leq 0$ for all $X^\dagger = X \in \mathcal{B}(\mathcal{H})$ and $t\geq 0$.*

6. Heisenberg picture

Consider now quantum dynamics in the Heisenberg picture, that is, $a_t := \Lambda_t^{\#}a$, where

$$\mathrm{tr}(\Lambda_t^{\#}a\cdot\rho) := \mathrm{tr}(a\cdot\Lambda_t\rho)\ . \tag{31}$$

Note, that Λ_t is CPT iff the dual map $\Lambda_t^{\#}$ is unital CP, i.e., $\Lambda_t^{\#}\mathbb{I}=\mathbb{I}$. Recall, that for any unital positive map Φ one has $||\Phi||=1$, where $||\Phi|| = \sup_{||a||=1}||\Phi(a)||$, and $||a||$ stands for an operator norm in $\mathcal{B}(\mathcal{H})$. It shows that $\Lambda_t^{\#}$ defines a family of contractions, that is,

$$||\Lambda_t^{\#}a|| \leq ||a||\ , \tag{32}$$

for any $a\in\mathcal{B}(\mathcal{H})$. Now, for Markovian dynamics one has

Proposition 5. *If Λ_t is Markovian then*

$$\frac{d}{dt}\,||\Lambda_t^{\#}a|| \leq 0\ , \tag{33}$$

for any $a\in\mathcal{B}(\mathcal{H})$, and $t\geq 0$.

Example. Consider once more the generator defined in (20). Note that $L_t^{\#} = L_t$. One has

$$||\Lambda_t\sigma_1|| = ||e^{-\Gamma(t)}\sigma_1|| = e^{-\Gamma(t)}||\sigma_1|| = e^{-\Gamma(t)}\ , \tag{34}$$

and hence

$$\frac{d}{dt}\,||\Lambda_t\sigma_1|| = -\dot{\Gamma}(t) = -\gamma(t)\ , \tag{35}$$

which shows that Markovianity of Λ_t implies $\gamma(t)\geq 0$.

Let us recall that if $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ is unital and 2-positive the following Kadison inequality holds

$$\Phi(aa^*) \geq \Phi(a)\Phi(a^*)\ . \tag{36}$$

This inequality may be used to characterize Markovian generators. Note, that Markovian dynamics $\Lambda_t^{\#}$ satisfies

$$\partial_t V_{t,s}^{\#} = V_{t,s}^{\#}L_t^{\#}\ ,\quad V_{s,s}^{\#} = \mathrm{id}\ , \tag{37}$$

where $V_{t,s}^{\#}$ denotes the dual propagator. Now, differentiating the Kadison inequality

$$V_{t,s}^{\#}(aa^*) \geq V_{t,s}^{\#}(a)V_{t,s}^{\#}(a^*)\ , \tag{38}$$

one finds

$$V_{t,s}^{\#}L_t^{\#}(aa^*) \geq V_{t,s}^{\#}L_t^{\#}(a)\cdot V_{t,s}^{\#}(a^*) + V_{t,s}^{\#}(a)\cdot V_{t,s}^{\#}L_t^{\#}(a^*)\ . \tag{39}$$

Taking $t = s$ and using $V^{\#}_{t,t} = \mathrm{id}$ one gets

$$L^{\#}_t(aa^*) \geq L^{\#}_t(a) \cdot a^* + a \cdot L^{\#}_t(a^*) \ . \tag{40}$$

Definition 4. A hermitian map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is dissipative iff

$$\Psi(aa^*) \geq \Psi(a) \cdot a^* + a \cdot \Psi(a^*) \ ,$$

for all $a \in \mathcal{B}(\mathcal{H})$. Ψ is completely dissipative if $\mathrm{id} \otimes \Psi$ is dissipative.

Actually, there is equivalent formulation of dissipativity of Ψ which generalizes classical result of Kolmogorov. One proves [5]

Proposition 6. *Let $\{P_1, \ldots, P_n\}$ be a family of orthogonal projectors $P_i P_j = P_i \delta_{ij}$ such that $P_1 + \cdots + P_n = \mathbb{I}$ in the Hilbert space $\mathcal{H}$. A hermitian map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is dissipative iff the real matrix $L_{ij} := \mathrm{tr}(P_i \Psi(P_j))$ satisfies the classical Kolmogorov conditions:*

$$L_{ij} \geq 0 \quad (i \neq j) \ , \quad \sum_{i=1}^{n} L_{ij} = 0 \ ,$$

for any $\{P_1, \ldots, P_n\}$.

Theorem 7. *Λ_t is Markovian iff $L^{\#}_t$ is completely dissipative.*

Corollary 8. *If Λ_t is CPT and $L^{\#}_t$ is not completely dissipative, then Λ_t represents non-Markovian evolution.*

7. Conclusions

We have analyzed the concept of (non)Markovianity of quantum evolution. One based on the divisibility property of the dynamical map and the other based upon the distinguishability of quantum states. It turns out that these two criteria do not coincide. However, they may be easily reconciled [19]. We provided the characterization of Markovian evolution in terms of the corresponding time-dependent local generator. Both Schrödinger and Heisenberg pictures are analyzed. The presentation is illustrated by simple example of qubit (2-level system) dynamics.

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Deformation Quantization of a Harmonic Oscillator in a General Non-commutative Phase Space: Energy Spectrum in Relevant Representations

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Abstract. In this paper, we discuss deformation quantization of a harmonic oscillator in a general non-commutative phase space, with both non-commuting spatial and momentum coordinates. Different representations are considered.

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Keywords. Deformation quantization, non-commutative phase space, harmonic oscillator, Landau problem, energy spectrum.

1. Introduction

In recent years, there is an increasing interest in the application of non-commutative (NC) geometry to physical problems [1] in solid-state and particle physics [2], mainly motivated by the idea of a strong connection of non-commutativity with field and string theories. Besides, the evidence coming from the latter and other approaches to the issues of quantum gravity suggests that attempts to unify gravity and quantum mechanics could ultimately lead to a non-commutative geometry of spacetime. The phase space of ordinary quantum mechanics is a well-known example of non-commuting space [3]. The momenta of a system in the presence of a magnetic field are non-commuting operators as well. Since the non-commutativity between spatial and time coordinates may lead to some problems with unitarity and causality, usually only spatial non-commutativity is considered. Besides, so far quantum theory on the NC space has been extensively studied, the main approach is based on the Weyl-Moyal correspondence which amounts to replacing the usual product by a $\star$-product in the NC space. Therefore, deformation quantization has special significance in the study of physical systems on the NC space. Moreover, the problem of quantum mechanics on NC spaces can be understood in the framework of deformation quantization [4, 5]. In the same vein, some works on harmonic oscillators (ho) in the NC space from the point of view of deformation quantization have been reported in [6, 7] and references therein.

In this paper, we consider different representations of a harmonic oscillator in a general full non-commutative phase space with both the spatial and momentum coordinates being non-commutative. Indeed, non-commutativity between momenta arises naturally as a consequence of non-commutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the non-commutative coordinates. This work continues the investigations stated in [6, 8] and [9] devoted to the study of a quantum exactly solvable D-dimensional NC oscillator with quasi-harmonic behavior. We intend to extend previous results presenting a similar analysis to the quantum version of the two-dimensional NC system with non-vanishing momentum components. For additional details on the motivation, see [6]. The physical model resembles the Landau problem in NC quantum mechanics extensively studied in the literature. See [10] and [11] and references therein for more details. Broadly put, it is worth noticing that the description of a system of a two-dimensional ho in a full 2D NC phase space is equivalent to that of the same ho in a constant magnetic field in some NC space.

2. Deformation Quantization (DQ) in NC phase space

Consider a $2D$ general NC phase space. The coordinates of position and momentum, $x = (x^1, x^2)$ and $p = (p^1, p^2)$, modeling the classical system of a two-dimensional ho maps into their respective quantum operators $\hat{x}$ and $\hat{p}$ giving rise to the Hamiltonian operator

$$\hat{H} = \frac{1}{2}\Big(\hat{p}_\mu\hat{p}^\mu + \hat{x}_\mu\hat{x}^\mu\Big) \tag{1}$$

with commutation relations

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar_{\text{eff}}\delta^{\mu\nu}, \;\; [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \;\; [\hat{p}^\mu, \hat{p}^\nu] = i\bar{\Theta}^{\mu\nu}, \quad \mu, \nu = 1, 2 \tag{2}$$

where $\Theta^{\mu\nu}$ and $\bar{\Theta}^{\mu\nu}$ are skew-symmetric tensors carrying the dimensions of $(\text{length})^2$ and $(\text{momentum})^2$, respectively. The effective Planck constant is given by

$$\hbar_{\text{eff}} = \hbar\Big(1 + \frac{\Theta^{\mu\nu}\bar{\Theta}^{\mu\nu}}{4D\hbar^2}\Big), \tag{3}$$

where $D = 2$ is the dimension of the NC space. The operators $\hat{x}^\mu$ and $\hat{p}^\nu$ can be rewritten as

$$\hat{p}^\mu = \hat{\pi}^\mu + \frac{1}{2\hbar}\bar{\Theta}^{\mu\nu}\hat{q}_\nu, \quad \hat{x}^\mu = \hat{q}^\mu - \frac{1}{2\hbar}\Theta^{\mu\nu}\hat{\pi}_\nu \tag{4}$$

in terms of $\hat{\pi}^\mu$ and $\hat{q}^\nu$ that obey the standard Weyl-Heisenberg algebra

$$[\hat{q}^\mu, \hat{\pi}^\nu] = i\hbar\delta^{\mu\nu}; \quad [\hat{q}^\mu, \hat{q}^\nu] = 0 = [\hat{\pi}^\mu, \hat{\pi}^\nu]. \tag{5}$$

In the deformation quantization theory of a classical system in the non-commutative space, one treats (x, p) and their functions as classical quantities, but replaces the ordinary product between these functions by the following generalized $\star$-product:

$$\begin{aligned}\star &= \star_{\hbar_{\text{eff}}} \star_{\Theta} \star_{\bar{\Theta}} \\ &= \exp\Big[\frac{i\hbar_{\text{eff}}}{2}\Big(\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{p^\mu} - \overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{x^\mu}\Big) + \frac{i\Theta^{\mu\nu}}{2}\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{x^\nu} + \frac{i\bar{\Theta}^{\mu\nu}}{2}\overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{p^\nu}\Big].\end{aligned} \tag{6}$$

The variables x^μ, p^μ on the NC phase space satisfy the following commutation relations similar to (2):

$$[x^\mu, p^\nu]_\star = i\hbar_{\text{eff}}\delta^{\mu\nu}, \;\; [x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}, \;\; [p^\mu, p^\nu]_\star = i\bar{\Theta}^{\mu\nu} \quad \mu, \nu = 1, 2 \tag{7}$$

with the following uncertainty relations:

$$\Delta x^1 \Delta x^2 \geq \frac{\Theta}{2} \qquad \Delta p^1 \Delta p^2 \geq \frac{\bar{\Theta}}{2} \tag{8}$$

$$\Delta x^1 \Delta p^1 \geq \frac{\hbar_{\text{eff}}}{2} \qquad \Delta x^2 \Delta p^2 \geq \frac{\hbar_{\text{eff}}}{2}. \tag{9}$$

The first two uncertainty relations show that measurements of positions and momenta in both directions x^1 and x^2 are not independent. Taking into account the fact that Θ and $\bar{\Theta}$ have dimensions of $(\text{length})^2$ and $(\text{momentum})^2$ respectively, then $\sqrt{\Theta}$ and $\sqrt{\bar{\Theta}}$ define fundamental scales of length and momentum which characterize the minimum uncertainties possible to achieve in measuring these quantities. One expects these fundamental scales to be related to the scale of the underlying field theory (possible the string scale), and thus to appear as small corrections at the low-energy level or quantum mechanics. Commonly, the time evolution function for a time-independent Hamiltonian H of a system is described by the $\star$-exponential function denoted here by $e_\star^{(\cdot)}$:

$$e_\star^{\frac{Ht}{i\hbar_{\text{eff}}}} := \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{t}{i\hbar_{\text{eff}}}\Big)^n \overbrace{H \star H \star \cdots \star H}^{\text{n times}}, \tag{10}$$

which is the solution of the following time-dependent Schrödinger equation

$$\begin{aligned} i\hbar_{\text{eff}}\frac{d}{dx}e_\star^{\frac{Ht}{i\hbar_{\text{eff}}}} &= H(x,p) \star e_\star^{\frac{Ht}{i\hbar_{\text{eff}}}} \\ &= H\Big(x^\mu + \frac{i\hbar_{\text{eff}}}{2}\partial_{p^\mu} + \frac{i\Theta^{\mu\rho}}{2}\partial_{x^\rho}, p^\nu - \frac{i\hbar_{\text{eff}}}{2}\partial_{x^\nu} + \frac{i\bar{\Theta}^{\mu\sigma}}{2}\partial_{x^\sigma}\Big)e_\star^{\frac{Ht}{i\hbar_{\text{eff}}}}. \end{aligned} \tag{11}$$

There corresponds the generalized $\star$-eigenvalue time-independent Schrödinger equation:

$$H \star \mathcal{W}_n = \mathcal{W}_n \star H = \mathcal{E}_n \mathcal{W}_n \tag{12}$$

where $\mathcal{W}_n$ and $\mathcal{E}_n$ stand for the Wigner function and the corresponding energy eigenvalue of the system. The Fourier-Dirichlet expansion for the time-evolution function defined as

$$e_\star^{\frac{Ht}{i\hbar_{\text{eff}}}} = \sum_{n=0}^{\infty} e^{\frac{-i\mathcal{E}_n t}{\hbar_{\text{eff}}}} \mathcal{W}_n \tag{13}$$

links the Wigner function to the $\star$-exponential function.

Provided the above, the operators on a NC Hilbert space can be represented by the functions on a NC phase space, where the operator product is replaced by relevant star-product. The algebra of functions of such non-commuting coordinates can be replaced by the algebra of functions on ordinary spacetime, equipped with a NC star-product. So, considering the transformations (4) and leaving out the operator symbol $\hat{}$, we arrive at (q, π) phase space and the commutation relations change into (5), with the star-product defined in the following way.

Definition 1. Let $C^\infty(\mathbb{R}^4)$ be the space of smooth functions $f : \mathbb{R}^4 \to \mathbb{C}$. For $f, g \in C^\infty(\mathbb{R}^4)$, the formal star product is defined by

$$f \star g = f \exp\left[\frac{i\hbar}{2}\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu\right] g. \tag{14}$$

Here the smooth functions f and g depend on the real variables q^1, q^2, π^1 and π^2, and

$$\begin{aligned}\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu &= \left(\frac{\overleftarrow{\partial}}{\partial q^1}, \frac{\overleftarrow{\partial}}{\partial \pi^1}, \frac{\overleftarrow{\partial}}{\partial q^2}, \frac{\overleftarrow{\partial}}{\partial \pi^2}\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\overrightarrow{\partial}}{\partial q^1} \\ \frac{\overrightarrow{\partial}}{\partial \pi^1} \\ \frac{\overrightarrow{\partial}}{\partial q^2} \\ \frac{\overrightarrow{\partial}}{\partial \pi^2} \end{pmatrix} \\ &= \frac{\overleftarrow{\partial}}{\partial q^1}\frac{\overrightarrow{\partial}}{\partial \pi^1} - \frac{\overleftarrow{\partial}}{\partial \pi^1}\frac{\overrightarrow{\partial}}{\partial q^1} + \frac{\overleftarrow{\partial}}{\partial q^2}\frac{\overrightarrow{\partial}}{\partial \pi^2} - \frac{\overleftarrow{\partial}}{\partial \pi^2}\frac{\overrightarrow{\partial}}{\partial q^2}.\end{aligned} \tag{15}$$

Therefore, the star product $f \star g$ represents a deformation of the classical product fg. This deformation depends on the Planck constant $\hbar$. In terms of physics, the difference $f \star g - fg$ describes quantum fluctuation depending on $\hbar$. For the present case,

$$q^\mu \star \pi^\nu - q^\mu \pi^\nu = \frac{i\hbar}{2}\delta^{\mu\nu}, \quad \pi^\nu \star q^\mu - \pi^\nu q^\mu = -\frac{i\hbar}{2}\delta^{\mu\nu}. \text{ Hence } [q^\mu, \pi^\nu]_\star = i\hbar\delta^{\mu\nu}. \tag{16}$$

Let us examine now the ho eigenvalue equation in different representations.

2.1. Harmonic oscillator eigenvalue equation in annihilation and creation operator representation

Building, in the standard manner, the creation and annihilation operators of ho system as

$$a_l = \frac{q^l + i\pi^l}{\sqrt{2}} \qquad \bar{a}_l = \frac{q^l - i\pi^l}{\sqrt{2}} \quad l = 1, 2 \tag{17}$$

and using the polar coordinates such that

$$q^l = \rho_l \cos\varphi_l, \qquad \pi^l = \rho_l \sin\varphi_l, \tag{18}$$

we solve the right and left eigenvalue equations

$$\begin{aligned} a_l \star f_{mn} &= \sqrt{m\hbar} f_{m-1,n} & \bar{a}_l \star f_{mn} &= \sqrt{(m+1)\hbar} f_{m+1,n} \\ f_{mn} \star a_l &= \sqrt{(n+1)\hbar} f_{m,n+1} & f_{mn} \star \bar{a}_l &= \sqrt{n\hbar} f_{m,n-1} \end{aligned} \tag{19}$$

to find the eigenfunctions f_{mn} as

$$f_{mn} \equiv 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i(n-m)\varphi_l} \Big(\frac{2\rho_l^2}{\hbar}\Big)^{\frac{n-m}{2}} L_m^{n-m}\Big(\frac{2\rho_l^2}{\hbar}\Big) e^{-\frac{\rho_l^2}{\hbar}}, \quad m,n \in \mathbb{N} \tag{20}$$

with

$$f_{00} = 2e^{-\rho_l^2/\hbar}. \tag{21}$$

L_m^{n-m} are the generalized Laguerre polynomials defined for $n = 0, 1, 2, \ldots$, $\alpha > 1$, by

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}. \tag{22}$$

Then the states defined by $b_{mn}^{(4)} = f_{m_1 n_1} f_{m_2 n_2}$, where $m = (m_1, m_2)$, $n = (n_1, n_2)$, $m_1, m_2, n_1, n_2 \in \mathbb{N}$, exactly solve the right and left eigenvalue problems of the Hamiltonian $H_0 = \sum_{l=1}^2 \bar{a}_l a_l$ as

$$H_0 \star b_{mn}^{(4)} = \hbar(|m|+1) b_{mn}^{(4)} \quad \text{and} \quad b_{mn}^{(4)} \star H_0 = \hbar(|n|+1) b_{mn}^{(4)} \tag{23}$$

where $|m| = m_1 + m_2$.

2.2. Harmonic oscillator eigenvalue equation in (q, π)-representation

Now, consider the Hamiltonian (1) and use the relation (5) to re-express it with the help of variables q and π as follows:

$$H = H_0 + H_L + H_q(\bar{\Theta}) + H_\pi(\Theta) \tag{24}$$

where

$$H_0 = \frac{1}{2}\Big((q^1)^2 + (q^2)^2 + (\pi^1)^2 + (\pi^2)^2\Big) \tag{25}$$

$$H_L = -\frac{\Theta + \bar{\Theta}}{2\hbar} \vec{q} \wedge \vec{\pi} \qquad \vec{q} \wedge \vec{\pi} = q^1 \pi_2 - q^2 \pi_1 \tag{26}$$

and

$$H_q(\bar{\Theta}) = \frac{\bar{\Theta}^2}{8\hbar^2}\Big((q^1)^2 + (q^2)^2\Big) \quad H_\pi(\Theta) = \frac{\Theta^2}{8\hbar^2}\Big((\pi^1)^2 + (\pi^2)^2\Big). \tag{27}$$

It is a matter of computation to verify that the Hamiltonians H_0 and H_L $\star$-commute. Idem for the Hamiltonians H_L and $H_I = H_q(\bar{\Theta}) + H_\pi(\Theta)$. Therefore, the Hamiltonians of family $\{H_0, H_L\}$, (respectively $\{H_L, H_I\}$) can be simultaneously measured. There follow two relevant situations.

2.2.1. Case $\Theta = -\bar{\Theta}$. The Hamiltonian H can be expressed as

$$H = \Big(1 + \frac{\Theta^2}{4\hbar^2}\Big) H_0 \tag{28}$$

and the states $b_{mn}^{(4)}$ solve the right and left eigenvalue problems of H as

$$H \star b_{mn}^{(4)} = \mathcal{E}_{m0}^R b_{mn}^{(4)} \qquad \mathcal{E}_{m0}^R = \hbar\big(1 + \big(\Theta^2/4\hbar^2\big)\big)(|m|+1) \tag{29}$$

and

$$b_{mn}^{(4)} \star H = \mathcal{E}_{0n}^L b_{mn}^{(4)} \qquad \mathcal{E}_{0n}^L = \hbar\big(1 + \big(\Theta^2/4\hbar^2\big)\big)(|n|+1) \tag{30}$$

where $m = (m_1, m_2)$, $n = (n_1, n_2)$, $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $|m| = m_1 + m_2$.

2.2.2. Case $\Theta = \bar{\Theta}$. The Hamiltonian H can be rewritten as

$$H = \Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)H_0 - \frac{\Theta}{\hbar}\overrightarrow{q} \wedge \overrightarrow{\pi}. \tag{31}$$

The eigenvectors of H_0 and H_L are eigenvectors of H, (as they commute each with other), with eigenvalues

$$\mathcal{E}^R_{mn} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|m| + 1) - (|n| - |m|)\Theta \tag{32}$$

and

$$\mathcal{E}^L_{mn} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|n| + 1) - (|m| - |n|)\Theta \tag{33}$$

corresponding to the right and left eigenvalue equations

$$H \star b^{(4)}_{mn} = \mathcal{E}^R_{mn} b^{(4)}_{mn} \tag{34}$$

and

$$b^{(4)}_{mn} \star H = \mathcal{E}^L_{mn} b^{(4)}_{mn}. \tag{35}$$

2.3. Harmonic oscillator eigenvalue equation in a general (q, π)-representation

The problem to be solved is equivalent to that of a two-dimensional Landau problem in a symmetric gauge on a non-commutative space. Indeed, the Hamiltonian H can be re-transcribed as

$$H = \frac{\alpha^2}{2}\Big((q^1)^2 + (q^2)^2\Big) + \frac{\beta^2}{2}\Big((\pi^1)^2 + (\pi^2)^2\Big) - \gamma\overrightarrow{q} \wedge \overrightarrow{\pi} =: H_0^\natural + H_L \tag{36}$$

where

$$\alpha^2 = 1 + \frac{\bar{\Theta}^2}{4\hbar^2}, \qquad \beta^2 = 1 + \frac{\Theta^2}{4\hbar^2}, \qquad \gamma = \frac{\Theta + \bar{\Theta}}{2\hbar} \tag{37}$$

Remark that the Hamiltonian terms $H_0^\natural$ and H_L commute. Therefore, the eigenvectors of $\{H_0^\natural, H_L\}$ are automatically eigenvectors of H. As matter of convenience, to solve the Schrödinger eigen-equation, let us choose the polar coordinates

$$q^1 = \rho\cos\varphi \qquad q^2 = \rho\sin\varphi \tag{38}$$

and assume the variable separability to write

$$\tilde{f}(\rho, \varphi) = \xi(\rho)e^{ik\varphi}, \quad k = 0, \pm 1, \pm 2, \cdots \tag{39}$$

Then, from the static Schrödinger equation on NC space, $H \star \tilde{f}(\rho, \varphi) = \mathcal{E}\tilde{f}(\rho, \varphi)$, we deduce the radial equation as follows:

$$\Big[-\frac{\hbar^2\beta^2}{2}\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big) + \frac{\alpha^2}{2}\rho^2 - \gamma\hbar k\Big]\xi(\rho, \varphi) = \mathcal{E}\xi(\rho, \varphi) \tag{40}$$

yielding the spectrum of H under the form

$$\mathcal{E} = \hbar\frac{\alpha^2}{\beta^2}(n + 1) - \hbar\gamma k, \quad n = 0, 1, 2, \ldots \tag{41}$$

with

$$\xi(\rho, \varphi) \propto e^{-\frac{\alpha}{\hbar\beta}\rho^2} H_n\left(\frac{\alpha}{\hbar\beta}\rho^2\right). \tag{42}$$

The last term of the energy spectrum $\mathcal{E}$ falls down when $\gamma = 0$, i.e., $\Theta = -\bar{\Theta}$. In this case, $\alpha^2 = \beta^2$ and we recover the discrete spectrum of the usual two-dimensional harmonic oscillator as expected. The results obtained here can be reduced to specific expressions reported in the literature [6] for particular cases. Besides, the formalism displayed in this work permits to avoid the appearance of infinite degeneracy of states observed when $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} = 0$ in [10] where the phase space is divided into two phases based on the following conditions on the deformation parameters:

- Phase I for $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} > 0$
- Phase II for $\hbar_{\text{eff}}^2 - \Theta\bar{\Theta} < 0$.

Finally, let us mention that the direct computation of the energy spectrum from the relation (24) instead of (36) introduces an unexpectable feature, i.e., the energy spectrum depends on the phase space variables as it should not be with respect to the study performed in [11]. Such a pathology is generated by the phase space variable dependence of the commutator

$$[H_0, H_I]_\star = i\frac{\Theta^2 - \bar{\Theta}^2}{4\hbar}(q^1\pi^1 + q^2\pi^2). \tag{43}$$

This could explain why previous investigations (see [6], [12] and [13] and references therein) were restricted to the cases $\Theta = \pm\bar{\Theta}$.

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Uniqueness Property for C^*-algebras Given by Relations with Circular Symmetry

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Abstract. A general method of investigation of the uniqueness property for C^*-algebra equipped with a circle gauge action is discussed. It unifies isomorphism theorems for various crossed products and Cuntz-Krieger uniqueness theorem for Cuntz-Krieger algebras.

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1. Introduction

The origins of C^*-theory and particularly the theory of universal C^*-algebras generated by operators that satisfy prescribed relations go back to the work of W. Heisenberg, M. Bohr and P. Jordan on matrix formulation of quantum mechanics, and among the most stimulating examples are algebras generated by anti-commutation relations and canonical commutation relations (in the Weyl form). The great advantage of relations of CAR and CCR type is *uniqueness of representation.* Namely, due to the celebrated Slawny's theorem, see, e.g., [1], the C^*-algebras generated by such relations are defined uniquely up to isomorphisms preserving the generators and relations. This *uniqueness property* is not only a strong mathematical tool but also has a significant physical meaning – if we had no such uniqueness, *different representations would yield different physics.*

The aim of the present note is to advertise a program of developing a general approach to investigation of uniqueness property and related problems based on exploring the symmetries of relations. We focus here, as a first attempt, on circular symmetries and propose a two-step method of investigation universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ generated by a set of generators $\mathcal{G}$ subject to relations $\mathcal{R}$ which could be schematically presented as follows:

$$\underset{\text{relations, circle action}}{(\mathcal{G},\mathcal{R},\{\gamma_\lambda\}_{\lambda\in\mathbb{T}})} \xrightarrow{\text{step 1}} \underset{\substack{\text{Hilbert bimodule}\\ \text{(reversible dynamics)}}}{(\mathcal{B}_0,\mathcal{B}_1)} \xrightarrow{\text{step 2}} \underset{\text{universal } C^*\text{-algebra}}{C^*(\mathcal{G},\mathcal{R}) = \mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}}$$

– we fix a circle gauge action $\gamma = \{\gamma\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$ which is induced by a circular symmetry in $(\mathcal{G},\mathcal{R})$; in the first step we associate to γ a non-commutative reversible dynamical system which is realized via a Hilbert bimodule $(\mathcal{B}_0,\mathcal{B}_1)$, and in the second step we use this system to determine the uniqueness property for $C^*(\mathcal{G},\mathcal{R})$.

2. Uniqueness property, universal C^*-algebras and gauge actions

Suppose we are given an abstract set of generators $\mathcal{G}$ and a set of *-algebraic relations $\mathcal{R}$ that we want to impose on $\mathcal{G}$. Formally $\mathcal{G}$ is a set and $\mathcal{R}$ is a set consisting of certain *-algebraic relations in a free non-unital *-algebra $\mathbb{F}$ generated by $\mathcal{G}$. By a *representation* of the pair $(\mathcal{G},\mathcal{R})$ we mean a set of bounded operators $\pi = \{\pi(g)\}_{g\in\mathcal{G}} \subset L(H)$ on a Hilbert space H satisfying the relations $\mathcal{R}$, and denote by $C^*(\pi)$ the C^*-algebra generated by $\pi(g)$, $g \in \mathcal{G}$. At this very beginning one faces the following two fundamental problems:

1. (*non-degeneracy problem*) Do there exists a *faithful representation* of $(\mathcal{G},\mathcal{R})$, i.e., a representation $\{\pi(g)\}_{g\in\mathcal{G}}$ of $(\mathcal{G},\mathcal{R})$ such that $\pi(g) \neq 0$ for all $g \in \mathcal{G}$?
2. (*uniqueness problem*) If one has two different faithful representation of $(\mathcal{G},\mathcal{R})$, do they generate isomorphic C^*-algebras? More precisely, does for any two faithful representations π_1, π_2 of $(\mathcal{G},\mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

   extends to the (necessarily unique) isomorphism $C^*(\pi_1) \cong C^*(\pi_2)$?

The first problem is important and interesting in its own rights, see [2], [3], however here we would like to focus on the second problem and thus throughout we assume that all the pairs $(\mathcal{G},\mathcal{R})$ under consideration are non-degenerate. We say that $(\mathcal{G},\mathcal{R})$ possess *uniqueness property* if the answer to question 2 is positive.

Any representation π of $(\mathcal{G},\mathcal{R})$ extends uniquely to a *-homomorphism, also denoted by π, from $\mathbb{F}$ into $L(H)$. The pair $(\mathcal{G},\mathcal{R})$ is said to be *admissible* if the function $|||\cdot||| : \mathbb{F} \to [0,\infty]$ given by

$$|||w||| = \sup\{\|\pi(w)\| : \pi \text{ is a representation of } (\mathcal{G},\mathcal{R})\}$$

is finite. Plainly, admissibility is a necessary condition for uniqueness property and therefore we make it our another standing assumption. Then the function $|||\cdot||| : \mathbb{F} \to [0,\infty)$ is a C^*-seminorm on $\mathbb{F}$ and its kernel

$$\mathbb{I} := \{w \in \mathbb{F} : |||w||| = 0\}$$

is a self-adjoint ideal in $\mathbb{F}$ – it is the smallest self-adjoint ideal in $\mathbb{F}$ such that the relations $\mathcal{R}$ become valid in the quotient $\mathbb{F}/\mathbb{I}$. We put

$$C^*(\mathcal{G},\mathcal{R}) := \overline{\mathbb{F}/\mathbb{I}}^{|||\cdot|||}$$

and call it a *universal C^*-algebra* generated by $\mathcal{G}$ subject to relations $\mathcal{R}$, cf. [4]. C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ is characterized by the property that any representation of $(\mathcal{G},\mathcal{R})$ extends uniquely to a representation of $C^*(\mathcal{G},\mathcal{R})$ and all representations of $C^*(\mathcal{G},\mathcal{R})$ arise in that manner. In particular, $(\mathcal{G},\mathcal{R})$ possess uniqueness property if and only if any faithful representation of $(\mathcal{G},\mathcal{R})$ extends to a faithful representation of $C^*(\mathcal{G},\mathcal{R})$.

3. Gauge actions – exploring the symmetries in the relations

We would like to identify the uniqueness property of $(\mathcal{G},\mathcal{R})$ by looking at the symmetries in $(\mathcal{G},\mathcal{R})$. In order to formalize this we use a natural torus action $\{\gamma_\lambda\}_{\lambda\in\mathbb{T}^{\mathcal{G}}}$ on $\mathbb{F}$ determined by the formula

$$\gamma_\lambda(g) = \lambda_g\, g, \qquad \text{for} \ \ g\in\mathcal{G} \ \ \text{and} \ \ \lambda = \{\lambda_h\}_{h\in\mathcal{G}} \in \mathbb{T}^{\mathcal{G}}$$

where $\mathbb{T} = \{z\in\mathbb{C} : |z| = 1\}$ is a unit circle. A closed subgroup $G\subset\mathbb{T}^{\mathcal{G}}$ may be considered as a *group of symmetries in the pair* $(\mathcal{G},\mathcal{R})$ if the restricted action $\gamma = \{\gamma_\lambda\}_{\lambda\in G}$ leaves invariant the ideal $\mathbb{I}$. Any such group gives rise to a point-wisely continuous group action on $C^*(\mathcal{G},\mathcal{R})$ and actions that arise in that manner are called *gauge actions.*

Let us from now on consider the case when $G\cong\mathbb{T}$, that is we have a circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$. Then for each $n\in\mathbb{Z}$ the formula

$$\mathcal{E}_n(b) := \int_{\mathbb{T}} \gamma_\lambda(b)\lambda^{-n}\, d\lambda$$

defines a projection $\mathcal{E}_n : C^*(\mathcal{G},\mathcal{R}) \to C^*(\mathcal{G},\mathcal{R})$, called *nth spectral projection*, onto the subspace

$$\mathcal{B}_n := \{b\in C^*(\mathcal{G},\mathcal{R}) : \gamma_\lambda(b) = \lambda^n b\}$$

called *nth spectral subspace* for γ, cf., e.g., [5]. Spectral subspaces specify a $\mathbb{Z}$-gradation on $C^*(\mathcal{G},\mathcal{R})$. Namely, $\bigoplus_{n\in\mathbb{Z}}\mathcal{B}_n$ is dense in $C^*(\mathcal{G},\mathcal{R})$, and

$$\mathcal{B}_n\mathcal{B}_m \subset \mathcal{B}_{n+m}, \ \ \mathcal{B}_n^* = \mathcal{B}_{-n} \ \ \text{for all} \ \ n,m\in\mathbb{Z}. \tag{1}$$

In particular, $\mathcal{B}_0$ is a C^*-algebra – the fixed point algebra for γ, and $\mathcal{E}_0 : \mathcal{B}\to\mathcal{B}_0$ is a conditional expectation. A circle action on a C^*-algebra $\mathcal{B}$ is called *semi-saturated* [5] if $\mathcal{B}$ is generated as a C^*-algebra by its first and zeroth spectral subspaces. We note that every continuous group endomorphism of $\mathbb{T}$ is of the form $\lambda\mapsto\lambda^n$, for certain $n\in\mathbb{Z}$, and hence it follows that $\mathcal{G}\subset\cup_{n\in\mathbb{Z}}\mathcal{B}_n$. In particular, we have

Lemma 1. *The circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$ on $C^*(\mathcal{G},\mathcal{R})$ is semi-saturated, that is $C^*(\mathcal{G},\mathcal{R}) = C^*(\mathcal{B}_0,\mathcal{B}_1)$ if and only if $\mathcal{G} = \mathcal{G}_0\cup\mathcal{G}_1$ for some disjoint sets $\mathcal{G}_0$, $\mathcal{G}_1$ and $\gamma_\lambda(g_0) = g_0$, $\gamma_\lambda(g_1) = \lambda g_1$, for all $g_i\in\mathcal{G}_i$.*

We introduce an important necessary condition for $(\mathcal{G},\mathcal{R})$ to possess uniqueness property.

Proposition 2. *The following conditions are equivalent:*

i) *each faithful representation of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of the fixed-point algebra $\mathcal{B}_0$.*

ii) *each faithful representation π of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of $C^*(\mathcal{G},\mathcal{R})$ if and only if there is a circle action β on $C^*(\pi)$ such that*

$$\beta_z(\pi(g)) = \pi(\gamma_z(g)), \qquad g \in \mathcal{G}.$$

Proof. i) $\Longrightarrow$ ii). It suffices to apply the gauge invariance uniqueness for circle actions, see, e.g., [5, 2.9] or [6, 4.2]. ii) $\Longrightarrow$ i). Assume that π is a faithful representation of $(\mathcal{G},\mathcal{R})$ such that its extension is not faithful on $\mathcal{B}_0$. The spaces $\{\pi(\mathcal{B}_n)\}_{n\in\mathbb{Z}}$ form a $\mathbb{Z}$-graded C^*-algebra and thus by [6, 4.2], there is a (unique) C^*-norm $\|\cdot\|_\beta$ on $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)$ such that the circle action β on $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)$ established by gradation extends onto the C^*-algebra $\mathcal{B} = \overline{\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n)}^{\|\cdot\|_\beta}$. Composing π with the embedding $\bigoplus_{n\in\mathbb{Z}}\pi(\mathcal{B}_n) \subset \mathcal{B}$ one gets a faithful representation π' of $(\mathcal{G},\mathcal{R})$ which is gauge-invariant but not faithful on $C^*(\mathcal{G},\mathcal{R})$. □

In the literature the statements showing that the condition ii) in Proposition 2 holds are often called *gauge-invariance uniqueness theorems* and therefore we shall say that the triple $(\mathcal{G},\mathcal{R},\gamma)$ has the *gauge-invariance uniqueness property* if each faithful representation of $(\mathcal{G},\mathcal{R})$ give rise to a faithful representation of the fixed-point algebra $\mathcal{B}_0$. In particular, this always holds for triples $(\mathcal{G},\mathcal{R},\gamma)$ such that $C^*(\mathcal{G},\mathcal{R})$ can be modeled as relative Cuntz-Pimsner algebra, see [3, Sect. 9] and sources cited there.

4. From relations to Hilbert bimodules

Let us fix a pair $(\mathcal{G},\mathcal{R})$ with a circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda\in\mathbb{T}}$. It follows from (1) that $\mathcal{B}_1$ can be naturally viewed as a *Hilbert bimodule* over $\mathcal{B}_0$, in the sense introduced in [7, 1.8]. Namely, $\mathcal{B}_1$ is a $\mathcal{B}_0$-bimodule with bimodule operations inherited from $C^*(\mathcal{G},\mathcal{R})$ and additionally is equipped with two $\mathcal{B}_0$-valued inner products

$$\langle a,b\rangle_R := a^*b, \qquad {}_L\langle a,b\rangle := ab^*$$

that satisfy the so-called imprimitivtiy condition: $a\cdot\langle b,c\rangle_R = {}_L\langle a,b\rangle\cdot c = ab^*c$, for all $a,b,c\in\mathcal{B}_1$. Thus we can consider *crossed product* $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z}$ of $\mathcal{B}_0$ by *the Hilbert bimodule* $\mathcal{B}_1$ constructed in [8], which could be alternatively defined as the universal C^*-algebra:

$$\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} = C^*(\mathcal{G}_\gamma,\mathcal{R}_\gamma)$$

where $\mathcal{G}_\gamma = \mathcal{B}_0 \cup \mathcal{B}_1$ and $\mathcal{R}_\gamma$ consists of all algebraic relations in the Hilbert bimodule $(\mathcal{B}_0,\mathcal{B}_1)$.

Proposition 3. *We have a natural embedding $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \hookrightarrow C^*(\mathcal{G}, \mathcal{R})$ which is an isomorphism if and only if γ is semi-saturated. Moreover, if γ is semi-saturated, then the following conditions are equivalent:*

i) $(\mathcal{G}, \mathcal{R})$ *possess uniqueness property*
ii) $(\mathcal{G}, \mathcal{R}, \gamma)$ *has gauge-invariance uniqueness property and* $(\mathcal{G}_\gamma, \mathcal{R}_\gamma)$ *possess uniqueness property*

Proof. Since the homomorphism $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \mapsto C^*(\mathcal{G}, \mathcal{R})$ is gauge-invariant and injective on $\mathcal{B}_0$ it is injective onto the whole $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z}$ by [5, 2.9]. The rest, in view of Proposition 2, is clear. □

The Hilbert bimodule $(\mathcal{B}_0, \mathcal{B}_1)$ is an imprimitivity bimodule (called also Morita-Rieffel equivalence bimodule), see [9], if and only if $\overline{\mathcal{B}_1^*\mathcal{B}_1} = \mathcal{B}_0$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*} = \mathcal{B}_0$. In general, $\overline{\mathcal{B}_1^*\mathcal{B}_1}$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*}$ are non-trivial ideals in $\mathcal{B}_0$ and we may treat $\mathcal{B}_1$ as a $\overline{\mathcal{B}_1\mathcal{B}_1^*} - \overline{\mathcal{B}_1^*\mathcal{B}_1}$-imprimitivity bimodule. This means, cf. [9, Cor. 3.33], that the induced representation functor

$$\widehat{h} = \mathcal{B}_1\text{-Ind}$$

is a homeomorphism $\widehat{h} : \widehat{\overline{\mathcal{B}_1^*\mathcal{B}_1}} \to \widehat{\overline{\mathcal{B}_1\mathcal{B}_1^*}}$ between the spectra of $\overline{\mathcal{B}_1^*\mathcal{B}_1}$ and $\overline{\mathcal{B}_1\mathcal{B}_1^*}$. Treating these spectra as open subsets of the spectrum $\widehat{\mathcal{B}}_0$ of $\mathcal{B}_0$ we may treat $\widehat{h}$ as a partial homeomorphism of $\widehat{\mathcal{B}}_0$. We shall say that $(\widehat{\mathcal{B}}, \widehat{h})$ is a *partial dynamical system dual to the bimodule* $(\mathcal{B}_0, \mathcal{B}_1)$. Partial homeomorphism $\widehat{h}$ is said to be *topologically free* if for each $n \in N$ the set of points in $\widehat{\mathcal{B}_0}$ for which the equality $\widehat{h}^n(x) = x$ (makes sense and) holds has empty interior.

Theorem 4 (main result). *Suppose that the partial homeomorphism $\widehat{h} = B_1$-Ind is topologically free. Then the pair $(\mathcal{G}_\gamma, \mathcal{R}_\gamma)$ possess uniqueness property and in particular*

i) *if $(\mathcal{G}, \mathcal{R}, \gamma)$ possess gauge-invariance uniqueness property, then any faithful representation of $(\mathcal{G}, \mathcal{R})$ extends to the faithful representation of $\mathcal{B}_1 \rtimes_{\mathcal{B}_0} \mathbb{Z} \subset C^*(\mathcal{G}, \mathcal{R})$.*
ii) *if γ is semi-saturated and $(\mathcal{G}, \mathcal{R}, \gamma)$ possess gauge-invariance uniqueness property, then $(\mathcal{G}, \mathcal{R})$ possess uniqueness property.*

Proof. Apply the main result of [10] and Proposition 3. □

5. Applications to crossed products and Cuntz-Krieger algebras

We show that our main result is a generalization of the so-called isomorphisms theorem for crossed products by automorphisms (see, for instance, [11, pp. 225, 226] for a brief survey of such results) by applying it to a crossed product by an endomorphisms which is considered to be one of the most successful constructions of this sort, see [12] and sources cited there. In particular, we shall use this crossed product to identify the uniqueness property for Cuntz-Krieger algebras.

5.1. Crossed products by monomorphisms with hereditary range

Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a monomorphism of a unital C^*-algebra $\mathcal{A}$. Let $\mathcal{G} = \mathcal{A} \cup \{S\}$ and let $\mathcal{R}$ consists of all *-algebraic relations in $\mathcal{A}$ plus the covariance relations

$$SaS^* = \alpha(a), \qquad S^*S = 1, \qquad a \in \mathcal{A}. \tag{2}$$

Then $C^*(\mathcal{G}, \mathcal{R}) \cong \mathcal{A} \rtimes_\alpha \mathbb{N}$ is the crossed product of $\mathcal{A}$ by α, which is equipped with a semi-saturated circle gauge action: $\gamma_\lambda(a) = a$, $\gamma_\lambda(S) = \lambda S$, $a \in \mathcal{A}$. Let us additionally assume that $\alpha(\mathcal{A})$ is a hereditary subalgebra of $\mathcal{A}$. This is equivalent to $\alpha(\mathcal{A}) = \alpha(1)\mathcal{A}\alpha(1)$. Then we have $S^*\mathcal{A}S \subset \mathcal{A}$ since for any $a \in \mathcal{A}$ there is $b \in \mathcal{A}$ such that $\alpha(b) = \alpha(1)a\alpha(1)$ and therefore

$$S^*aS = S^*\alpha(1)a\alpha(1)S = S^*\alpha(b)S = S^*SbS^*S = b \in \mathcal{A}.$$

Hence on one hand $\mathcal{A} = \mathcal{B}_0$ is the fixed point algebra for γ and $\mathcal{B}_1 = \mathcal{B}_0 S$ is the first spectral subspace. On the other hand the spectrum of the hereditary subalgebra $\alpha(\mathcal{A})$ may be naturally identified with an open subset of $\widehat{\mathcal{A}}$, see, e.g., [13, Thm. 5.5.5], and then the dual $\widehat{\alpha} : \widehat{\alpha(\mathcal{A})} \to \widehat{\mathcal{A}}$ to the isomorphism $\alpha : \mathcal{A} \to \alpha(\mathcal{A})$ becomes a partial homeomorphism of $\widehat{\mathcal{A}}$. Under this identification one gets

$$\widehat{\alpha} = \mathcal{B}_1\text{-Ind}$$

and hence if the partial system $(\widehat{\mathcal{A}}, \widehat{\alpha})$ dual to $(\mathcal{A}, \alpha)$ is topologically free, then $(\mathcal{G}, \mathcal{R})$ possess uniqueness property.

5.2. Cuntz-Krieger algebras

Let $\mathcal{G} = \{S_i : i = 1, \dots, n\}$, where $n \geq 2$, and let $\mathcal{R}$ consists of the Cuntz-Krieger relations

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*, \qquad S_i^* S_k = \delta_{i,k} S_i^* S_i, \qquad i,k = 1, \dots, n, \tag{3}$$

where $\{A(i,j)\}$ is a given $n \times n$ zero-one matrix such that every row and every column of A is non-zero, and $\delta_{i,j}$ is Kronecker symbol. Then $C^*(\mathcal{G}, \mathcal{R})$ is the Cuntz-Krieger algebra $\mathcal{O}_A$ and the celebrated Cuntz-Krieger uniqueness theorem, cf. [14, Thm. 2.13], states that the pair $(\mathcal{G}, \mathcal{R})$ possess uniqueness property if and only if the so-called *condition* (I) holds:

- **(I)** the space $X_A := \{(x_1, x_2, \dots) \in \{1, \dots, n\}^{\mathbb{N}} : A(x_k, x_{k+1}) = 1\}$ has no isolated points (considered with the product topology)

We may recover this result applying our method to the standard circle gauge action on $\mathcal{O}_A$ determined by equations $\gamma_\lambda(S_i) = \lambda S_i$, $i = 1, \dots, n$. Indeed, the fixed point C^*-algebra for γ coincides with the so-called AF-core

$$\mathcal{F}_A = \overline{\text{span}}\{S_\mu S_\nu^* : |\mu| = |\nu| = k,\ k = 1, \dots\}$$

where for a multiindex $\mu = (i_1, \dots, i_k)$, with $i_j \in 1, \dots, n$, we denote by $|\mu|$ the length k of μ and write $S_\mu = S_{i_1} S_{i_2} \cdots S_{i_k}$. Moreover, any faithful representation of the Cuntz-Krieger relations (3) yields a faithful representation of $\mathcal{F}_A$, that is $(\mathcal{G},\mathcal{R},\gamma)$ possess gauge-invariance uniqueness property. Following [12] we put

$$S := \sum_{i,j} \frac{1}{\sqrt{n_j}}\, S_i P_j$$

where $n_j = \sum_{i=1}^n A(i,j)$ and $P_j = S_j S_j^*$, $j = 1,\dots,n$. A routine computation shows that $S\mathcal{F}_A S^* \subset \mathcal{F}_A$, $S^*\mathcal{F}_A S \subset \mathcal{F}_A$ and $S^*S = 1$ (S is an isometry). Hence the mapping $\mathcal{F}_A \ni a \mapsto \alpha(a) := SaS^* \in \mathcal{F}_A$ is a monomorphism with a hereditary range. It is uniquely determined by the formula

$$\alpha\Big(S_{i_2\mu}S^*_{j_2\nu}\Big) = \frac{1}{\sqrt{n_{i_2}n_{j_2}}}\sum_{i,j=1}^{n} S_{i\,i_2\mu}S^*_{j\,j_2\nu}. \tag{4}$$

From the construction any representation of relations (3) yields a representation of $(\mathcal{F}_A,\alpha)$ as introduced in the previous subsection. Conversely, if S satisfies (2) where $\mathcal{A} = \mathcal{F}_A$, then one gets representation of (3) by putting $S_i := \sum_{j=1}^n A(i,j)\sqrt{n_j}P_i S P_j$. Thus we have a natural isomorphism

$$\mathcal{O}_A \cong \mathcal{F}_A \rtimes_\alpha \mathbb{N}$$

under which the introduced gauge actions coincide. Hence we may identify the partial dynamical system dual to the Hilbert bimodule $(\mathcal{B}_1,\mathcal{B}_0)$ where $\mathcal{B}_0 = \mathcal{F}_A$ and $\mathcal{B}_1 = \mathcal{F}_A S$ with the partial dynamical system $(\widehat{\mathcal{F}_A},\widehat{\alpha})$ dual to $(\mathcal{F}_A,\alpha)$, as introduced in the previous subsection.

In order to identify the topological freeness of $\widehat{\alpha}$ we define $\pi_\mu \in \widehat{\mathcal{A}}$ for any infinite path $\mu = (i_1, i_2, \dots)$, $A(i_j, i_{j+1}) = 1$, $j \in \mathbb{N}$, to be the GNS-representation associated to the pure state $\omega_\mu : \mathcal{F}_A \to \mathbb{C}$ determined by the formula

$$\omega_\mu(S_\nu S_\eta^*) = \begin{cases} 1 & \nu = \eta = (\mu_1,\dots,\mu_n) \\ 0 & \text{otherwise}\end{cases} \qquad \text{for} \quad |\nu| = |\eta| = n. \tag{5}$$

Using description of the ideal structure in $\mathcal{F}_A$ in terms of Bratteli diagrams [15], similarly as in [10], one can show that representations π_μ form a dense subset of $\widehat{\mathcal{F}_A}$ and

$$\widehat{\alpha}(\pi_{(\mu_1,\mu_2,\mu_3,\dots)}) = \pi_{(\mu_2,\mu_3,\dots)}, \qquad \text{for any } (\mu_1,\mu_2,\mu_3,\dots).$$

In particular, it follows that *topological freeness of* $\widehat{\alpha}$ *is equivalent to condition* (I). Accordingly

> *our main result, Theorem 4, when applied to Cuntz-Krieger relations is equivalent to the Cuntz-Krieger uniqueness theorem.*

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On Maximal $\mathbb{R}$-split Tori Invariant under an Involution

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Abstract. Symmetric k-varieties have been a topic of interest in several fields of mathematics and physics since the 1980's. For $k = \mathbb{R}$, symmetric $\mathbb{R}$-varieties are commonly called real symmetric spaces; however, the generalization over other fields play a role in the study of arithmetic subgroups, geometry, singularity theory, Harish Chandra modules and most importantly representation theory of Lie groups.

The preliminary study of the rationality properties of these spaces over various base fields was published by Helminck and Wang [1]. In order to study the representations associated with these symmetric k-varieties one needs a thorough understanding of the orbits of parabolic k-subgroups, P_k, acting on the symmetric k-varieties, G_k/H_k. This paper's contribution is the classification of the orbits of $P \setminus G/H$ which are determined by the H-conjugacy classes of σ-stable maximal quasi k-split tori.

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1. Introduction and notation

Symmetric k-varieties are the homogeneous spaces defined G_k/H_k where G_k and H_k are the k-points of a reductive group G and H, the fixed point group of some involution. They play a role in geometry, singularity theory, and the cohomology of arithmetic groups. However, they are probably best known for their role in representation theory. The first breakthrough was made when Harish-Chandra commenced his study of general semisimple Lie groups, which finally led to the Plancherel formula. The next step was to study the representation theory of the general semisimple symmetric spaces which has been considered by Brylinski, Delorme, Carmona, Matsuki, Oshima, Schlichtkrull, van der Ban and many others.

The orbits of parabolic k-subgroups acting on a symmetric k-variety are of fundamental importance in the study of induced representations. The characterization of these orbits involves conjugacy classes of σ-stable maximal k-split tori and for each of these σ-stable maximal k-split tori a quotient of Weyl groups.

There are descriptions of some of these orbit decompositions in [1], the focus is on the orbits of parabolic k-subgroups acting on a variety, $P_k \setminus G_k/H_k$. Such a decomposition can be characterized as the P_k-orbits action on G_k/H_k, the H_k-orbits on $P_k \setminus G_k$ or the orbits of $P_k \times H_k$ on G. While these orbits are characterized for any field k the actually classification requires first the classification of orbit decompositions of the related $P \setminus G/H$. There exists a map between the orbits of $P_k \setminus G_k/H_k$ onto orbits of $P \setminus G/H$. After classifying the orbits of the latter one determines the fibers of the representatives and find the classification of the former. This paper's will discuss the classification of the orbits of $P \setminus G/H$ which are determined by the H-conjugacy classes of σ-stable maximal quasi k-split tori; however, there are 171 cases to consider and the! classification is quite long. Please see [2] for the full classification.

Helminck and Wang described the double cosets as follows:

Theorem 1 ([1, Proposition 6.10]). *Let $\{A_i \mid i \in I\}$ be representatives of the H_k-conjugacy classes of σ-stable maximal k-split tori in G. Then*

$$P_k \setminus G_k/H_k \cong \bigcup_{i \in I} W_{G_k}(A_i)/W_{H_k}(A_i).$$

The goal will be to explicitly determine the set I for $k = \mathbb{R}$ in order to calculate the Weyl groups, $W_{G_k}(A_i)$ and $W_{H_k}(A_i)$.

1.1. Notation

Definition 1. A torus, T, is called σ-stable if $\sigma(T) = T$. Then $T = T_\sigma^+ T_\sigma^-$, where

$$T_\sigma^+ = (T \cap H)^0 \text{ and } T_\sigma^- = \{x \in T | \sigma(x) = x^{-1}\}^0$$

A torus, A, is called σ-split if $\sigma(a) = a^{-1}$ for all $a \in A$. A quasi k-split torus is a torus that is G-conjugate to a k-split torus. Last, a torus, S, is called σ-fixed if $\sigma(s) = s$ for all $s \in S$. Note, a (σ, k)-split torus is both σ-split and k-split. Let $\mathfrak{A}_k^{(\theta,\sigma)}$ be the set of all (θ, σ)-stable maximal k-split tori. $\mathfrak{A}^{(\theta,\sigma)}$ be the set of (θ, σ)-stable maximal quasi k-split tori. Also, $\mathfrak{A}_0^{(\theta,\sigma)}$ be the set of quasi k-split tori that are H-conjugate with a k-split torus.

Since we will be looking at the H_k^+ or H-conjugacy classes of these various sets, we will denote these classes by: $\mathfrak{A}_k^{(\sigma,\theta)}/H_k^+$, $\mathfrak{A}^{(\theta,\sigma)}/H$, and $\mathfrak{A}_0^{(\theta,\sigma)}/H$, respectively.

We will call $\Phi(A) = \Phi_\theta$ the root system of a torus θ-split torus A with associated Weyl group $W(A)$. In general, the Weyl group of a torus, T, will be $W(T, L_k) = W_{L_k}(T) = N_{L_k}(T)/Z_{L_k}(T)$, where

$$N_{L_k}(T) = \{x \in L_k \mid xTx^{-1} \subset T\},$$
$$Z_{L_k}(T) = \{x \in L_k \mid xt = tx \text{ for all } t \in T\}.$$

We will also be looking at $\Phi_{\theta,\sigma} = \Phi(A, A_\sigma^-) = \Phi(A) \cap \Phi(A_\sigma^-)$. For $w \in W(A)$, $\Phi(w) = \{\alpha \in \Phi(A) \mid w(\alpha) = -\alpha\}$.

The following sections will highlight important portions of the final classification. The goal is to determine the H_k-conjugacy classes of maximal $\mathbb{R}$-split tori for the orbit decomposition $P_{\mathbb{R}} \setminus G_{\mathbb{R}}/H_{\mathbb{R}}$. The following steps will be discussed.

1. A Cartan involution, θ, commuting with σ will convert the problem into a pair, (θ, σ), of commuting involutions over $\mathbb{C}$ while simplifying the $\mathbb{R}$-split requirement. One involution over $\mathbb{R}$ becomes a pair of commuting involutions over $\mathbb{C}$.
2. All tori can be put into standard position and each torus can be associated with a Weyl group element.
3. Classify the H-conjugacy classes of σ-stable maximal quasi $\mathbb{R}$-split tori on route to the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.
4. Employ the use of the associated pair $(\theta, \sigma\theta)$ and classify the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.

This paper will demonstrate 1. through 3. and end with a description of associated pairs and the role played to determine 4. My current research is to complete the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori.

2. Cartan involutions

Definition 2. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the decomposition into the $+1$ and -1-eigenspaces of θ. Then $\theta \in \mathrm{Aut}(\mathfrak{g}_0)$ is called a Cartan involution if $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$. A subalgebra be called compact if the Killing form restricted to $\mathfrak{k}_0$ is negative definite.

The Cartan involution plays an important role, when $k = \mathbb{R}$, in the classification of the representatives of the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal $\mathbb{R}$-split tori. A Cartan involution, θ, commuting with σ will simplify the into a pair, (θ, σ), of commuting involutions over $\mathbb{C}$ while simplifying the $\mathbb{R}$-split requirement. This changes the problem from one involution to commuting involutions over $\mathbb{C}$.

In our discussion, we have a fixed involution σ and can find a Cartan involution that will commute with σ.

Theorem 2 ([3, Lemma 10.2]). *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, θ a Cartan involution, and σ any involution. Then there exists $\phi \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\phi\theta\phi^{-1}$ commutes with σ.*

Theorem 3 ([4, Theorem 10.6]). *The inner isomorphism classes of semisimple locally symmetric pairs $(\mathfrak{g}_0, \mathfrak{h})$ correspond bijectively to the inner isomorphism classes of ordered pairs of commuting involutions (θ, σ) of $\mathfrak{g}$ or $\mathrm{Aut}(\mathfrak{g})^0$. The outer isomorphism classes correspond bijectively as well.*

For $k = \mathbb{R}$ one studies the structure of real reductive algebraic groups in the complex case with a pair of commuting involutions (where one is a Cartan involution) instead of one involution of a real reductive algebraic group.

Let θ be a Cartan involution of G over k and σ a k-involution with $\sigma\theta = \theta\sigma$. Consider the following propositions from [1]:

1. (Proposition 11.18) Given any σ-stable maximal k-split torus A of G, there is a $h \in H_k$ such that hAh^{-1} is θ-stable.
2. Any θ-stable k-split torus is θ-split.
3. (Lemma 11.5) Any maximal θ-split k torus of G is maximal (θ, k)-split.

Therefore, any σ-stable maximal $\mathbb{R}$-split torus of G can be viewed as a (σ, θ)-stable maximal $\mathbb{R}$-split torus (or θ-split torus) of G. An important corollary follows from Theorem 1 when using this relation.

Corollary 4 ([1, Corollary 12.11]). *Let K be the fixed point group of θ, H a k-open subgroup of the fixed point group of σ and $H^+ = H \cap K$. Then*

$$P_k \backslash G_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i) / W_{H_k^+}(A_i)$$

where $\{A_i \mid i \in I\}$ are the representatives of the H_k^+-conjugacy classes of (σ, θ)-stable maximal k-split tori in G.

In fact, pairs of commuting involutions over complex groups were classified in [4]. The notation from that paper will used to represent involutions through this section and next. Each involution has a Cartan type and each type has a diagram representation. From these diagrams, which were created using an ordered basis, one determines the type of the maximal $\mathbb{R}$-split (θ-split) torus (Φ_θ with basis Δ_θ) and the σ-split torus in the maximal $\mathbb{R}$-split ($\Phi_{\sigma,\theta}$ with basis $\Delta_{\sigma,\theta}$) for each pair of commuting involutions. There are 171 irreducible pairs to consider. Knowing the type and dimension of the maximal $(\sigma, \mathbb{R})$-split torus is necessary for the classification.

3. Characterizing standard involutions

As seen in the previous section, we can find the type and dimension of the maximal $(\sigma, \mathbb{R})$-split torus in the set $\mathfrak{A}_k^{(\theta,\sigma)}$.

3.1. Standard position

Definition 3. For A_1, $A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$, the pair (A_1, A_2) is called standard if $A_1^- \subset A_2^-$ and $A_1^+ \supset A_2^+$. We say that A_1 is standard with respect to A_2.

Theorem 5 ([5, Theorem 3.6]). *Let (A_1, A_2) be a standard pair of (θ, σ)-stable $\mathbb{R}$-split (or quasi $\mathbb{R}$-split) tori of G. Then the following hold:*

1. *There exists $g \in Z(A_1^- A_2^+)$ such that $gA_1g^{-1} = A_2$.*
2. *If $n_1 = g^{-1}\sigma(g)$ and $n_2 = \sigma(g)g^{-1}$, then $n_1 \in N(A_1)$ and $n_2 \in N(A_2)$.*
3. *Let w_1 and w_2 be the images of n_1 and n_2 in $W(A_1)$ and $W(A_2)$ respectively. Then $w_1^2 = e, w_2^2 = e$, and $(A_1)^+_{w_1} = (A_2)^+_{w_2} = A_1^- A_2^+$ which characterizes w_1 and w_2.*

Corollary 6. *Fix an element* $A \in \mathfrak{A}_k^{(\theta,\sigma)}$ *such that* A_σ^- *is maximal. Let* A_1 *be put in standard position with* A *where* A^- *is a maximal* $(\sigma, \mathbb{R})$*-split torus of* G *Then the following hold:*

1. *There exists* $g \in Z(A_1^- A^+)$ *such that* $gA_1g^{-1} = A$.
2. *If* $n = \sigma(g)g^{-1}$, *then* $n \in N(A)$.
3. *Let* w *be the image of* n *in* $W(A)$. *Then* $w^2 = e$, *and* $(A)_w^+ = A_1^- A^+$ *which characterizes* w.

For any tori $A_1, A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$ put in standard position with A, there is an associated element in $W(A)$. Each has an element g which is associated with an $n \in N(A)$ whose image in $W(A)$ is w. These images w_1 and w_2 are called the A_1-standard and A_2-standard involutions, respectively.

Let w_1 and w_2 be the A_1-standard and A_2-standard involutions, respectively, in $W(A)$. Now, we can discuss the tori based on these elements of the finite Weyl group.

Proposition 7 ([1, Proposition 12.6]). *Assume that* $A_1, A_2 \in \mathfrak{A}_k^{(\theta,\sigma)}$ *are both standard with respect to* A. *Let* w_1 *and* w_2 *be the* A_1*-standard and* A_2*-standard involutions, respectively, in* $W(A)$. *Then* A_1 *and* A_2 *are* H_k^+*-conjugate if and only if* w_1 *and* w_2 *are conjugate under* $W(A, H_k^+)$

Corollary 8. *Assume that* A_1', $A_2' \in \mathfrak{A}^{(\theta,\sigma)}$ *are both standard with respect to* A. *Let* w_1' *and* w_2' *be the* A_1'*-standard and* A_2'*-standard involutions, respectively, in* $W(A)$. *Then* A_1' *and* A_2' *are* H*-conjugate if and only if* w_1 *and* w_2 *are conjugate under* $W(A, H)$.

3.2. Singular involutions

What remains is to determine which involutions in $W(A)$ are A_i-standard involutions for some $A_i \in \mathfrak{A}^{(\theta,\sigma)}$ or $\mathfrak{A}_k^{(\theta,\sigma)}$.

Definition 4. Let $A \in \mathfrak{A}^{(\sigma,\theta)}$, $w \in W(A)$ and $G_w = Z(A_w^+)$. w is called σ-singular when following properties hold.

1. $w^2 = e$.
2. $\sigma w = w\sigma$.
3. $\sigma|[G_w, G_w]$ is k-split.

w is called (θ, σ)-singular if w if σ-singular and $\sigma\theta|[G_w, G_w]$ is k-split. A root $\alpha \in \Phi(A)$ is called σ-singular ((θ, σ)-singular) if the corresponding reflection $s_\alpha \in W(A)$ is σ-singular ((θ, σ)-singular).

Proposition 9. *An involution* $w \in W(A)$ *is a* σ*-singular* ((σ, θ)*-singular*) *involution iff* w *is an* A_i*-standard involution for some* $A_i \in \mathfrak{A}^{(\theta,\sigma)}$ ($\mathfrak{A}_k^{(\sigma,\theta)}$).

Proposition 10. *Let* $A \in \mathfrak{A}^{(\theta,\sigma)}$ ($\mathfrak{A}_k^{(\theta,\sigma)}$) *with* A_σ^- *maximal. Then there is a one-to-one correspondence between the* $W(A, H)$*-*($W(A, H_k^+)$)*-conjugacy classes of* A_i*-standard involutions in* $W(A)$ *and the* $W(A, H)$*-*($W(A, H_k^+)$)*-conjugacy classes of* σ*-singular* ((θ, σ)*-singular*) *involutions in* $W(A)$.

Now the goal is to classify the singular involutions in $W(A)$. A complete discussion of conjugacy classes of elements in the Weyl group can be found in [5]. In summary, let $\Phi(A)$ be irreducible and $w \in W(A)$ an involution, then $\Phi(w)$ is of type $r \cdot A_1 + X_\ell$, where either $X_\ell = \emptyset$ or one of $B_\ell(\ell \geq 1)$, $C_\ell(\ell \geq 1)$, $D_\ell(\ell \geq 1)$, E_7, E_8, F_4, or G_2, where $r \cdot A_1 = A_1 + A_1 + \cdots + A_1$ r times.

Let $\mathfrak{W}$ be the set of all W-conjugacy classes of involutions in W. If we define an order $>$ on $\mathfrak{W}$ then for $[w_1], [w_2] \in \mathfrak{W}$ we have $[w_1] > [w_2]$ if and only if $\Delta(w_1) \subset \Delta(w_2)$ for some representatives w_i of $[w_i](i = 1, 2)$.

One builds diagrams of these conjugacy classes as seen in [6]. Once the A_i-standard involutions in $W(A)$ are identified the diagram describes the conjugacy classes and types of tori. If $w_1, w_2 \in W(A)$ are A_1 and A_2-standard involutions of A_1 and A_2 then

$$A_1^- \subset A_2^- \iff A_{w_1}^- \supset A_{w_2}^-.$$

Hence,

$$[A_1] < [A_2] \iff [w_1] < [w_2].$$

Example. Suppose Φ_θ is of type B_3, let $\Delta_\theta = \{\alpha_1, \alpha_2, \alpha_3\}$ be a basis for Φ_θ. Then $\Phi(w)$ is some subset of Φ_θ. The following list describes possible types of the basis for $\Phi(w)$, $\Delta(w)$. We use the notation B_1 to designate the unique shortest root of type A_1.

- Type $\Delta(w) = $ empty.
- Type $\Delta(w) = A_1$.
- Type $\Delta(w) = B_1$.
- Type $\Delta(w) = 2 \cdot A_1$.
- Type $\Delta(w) = B_2$.
- Type $\Delta(w) = B_3$.

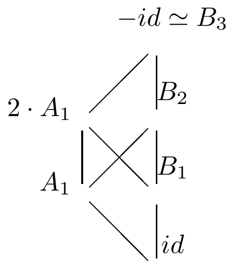


Figure 1. Conjugacy classes of involutions in Weyl group of Φ type B_3

4. H-conjugacy classes of $\mathfrak{A}^{(\theta,\sigma)}$

Proposition 11. *$\alpha \in \Phi(A)$ is a σ-singular root if and only if $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$.*

Lemma 12 ([5, Theorem 4.6]). *Let A be a (θ, σ)-stable $\mathbb{R}$-split torus of G with A_σ^- a maximal $(\sigma, \mathbb{R})$-split torus of G and $w \in W(A)$, $w^2 = e$. Then the following are equivalent:*

1. *w is σ-singular.*
2. *$A_w^- \subset A_\sigma^-$.*

Proof. ($\Longrightarrow$) α is a σ-singular root then by Lemma 12, $A_{s_\alpha} \subset A_\sigma^-$. Therefore, $\alpha \in \Phi(A_\sigma^-)$. Since $\alpha \in \Phi(A)$ then $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$.

($\Longleftarrow$) $\alpha \in \Phi(A) \cap \Phi(A_\sigma^-)$, then $\alpha \in \Phi(A)$ and $w = s_\alpha \in W(A)$ so $w^2 = e$. Since $\alpha \in \Phi(A_\sigma^-)$, $A_{s_\alpha}^- \subset A_\sigma^-$. By Lemma 12, s_α is σ-singular and α is a σ-singular root. □

Theorem 13. *Let $A \in \mathfrak{A}^{(\theta,\sigma)}$ with A_σ^- maximal. Then there is a one-to-one correspondence between the $W(A)$-conjugacy classes of σ-singular involutions in $W(A)$ and the $W(A)$-conjugacy classes of elements in $W(A, A_\sigma^-)$ where $W(A, A_\sigma^-)$ is the Weyl group of $\Phi(A, A_\sigma^-) = \Phi(A) \cap \Phi(A_\sigma^-)$.*

Example.

Ex.	Type (θ, σ)	Type Φ_θ $\Phi(A)$	Type $\Phi_{\sigma,\theta} \cap \Phi_\theta$ $\Phi(A, A_\sigma^-)$	max. involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$
(1)	$A_{2\ell+1}^{2\ell+1,\ell}(\mathrm{I}, \mathrm{II})$	$A_{2\ell+1}$	$\emptyset$	id
(2)	$A_{4\ell-1}^{2\ell,2\ell-1,\epsilon_0}(\mathrm{III_b}, \mathrm{II})$	$C_{2\ell}$	$\ell \cdot A_1$	$\ell \cdot A_1$
$\ell = 2$	$A_7^{4,3}(\mathrm{III_b}, \mathrm{II}, \epsilon_0)$	C_4	$2 \cdot A_1$	$2 \cdot A_1$
(3)	$B_\ell^{q,p}(\mathrm{I_a}, \mathrm{I_a}, \epsilon_\mathrm{i})$	B_q	B_p	B_p
$\ell = 5$	$B_5^{4,3}(\mathrm{I_a}, \mathrm{I_a}, \epsilon_\mathrm{i})$	B_4	B_3	B_3

Table 1

In Ex. (1), $\Phi(A, A_\sigma^-) = \emptyset$ and $W(A, A_\sigma^-) = \text{id}$. There is only one $W(A)$-conjugacy class of σ-singular roots; therefore, there is only one H-conjugacy class (σ, θ)-stable maximal quasi $\mathbb{R}$-split tori. In Ex. (2), seen in Figure 2, there is only

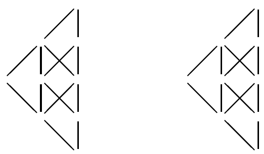


Figure 2. $\mathfrak{A}^{(\theta,\sigma)}/H$ for Ex. (2) & Ex. (3)

one $W(A)$-conjugacy class of σ-singular roots at each dimension; therefore, there is only one H-conjugacy class (σ, θ)-stable maximal quasi $\mathbb{R}$-split tori for each dimension. Last, in Ex. (3), as seen in Figure 2, one sees the $W(A)$-conjugacy class of σ-singular roots at each dimension from the diagram. Also, ones count the H-conjugacy classes (σ, θ)-stable maximal quasi $\mathbb{R}$-split tori for each dimension.

The complete classification of $\mathfrak{A}^{(\theta,\sigma)}/H$ in all 171 cases is quite long and can be found in [2]. This classification will help to determine the $H_{\mathbb{R}}$-conjugacy classes of (θ, σ)-stable maximal $\mathbb{R}$-split tori.

5. H_k-conjugacy classes of (θ, σ)-stable maximal k-split tori

The classification of the $H_{\mathbb{R}}$-conjugacy classes of σ-stable maximal tori can be simplified into a classification of objects in the Weyl group. However, determining the (θ, σ)-singular involutions and the appropriate conjugacy classes requires deeper investigation.

5.1. Associated Pairs

$$\begin{array}{ccccc} (\mathfrak{g}, \mathfrak{h}) & & (\mathfrak{g}, \mathfrak{h}^a) & & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \\ (\theta, \sigma) & \leftarrow \text{associated} \rightarrow & (\theta, \sigma\theta) & \leftarrow \text{dual} \rightarrow & (\sigma\theta, \theta) \\ \uparrow & & & & \uparrow \\ \text{dual} & & & & \text{associated} \\ \downarrow & & & & \downarrow \\ (\mathfrak{g}^d, \mathfrak{h}^d) & & (\mathfrak{g}^d, \mathfrak{h}^a) & & (\mathfrak{g}^{ad}, \mathfrak{h}) \\ (\sigma, \theta) & \leftarrow \text{associated} \rightarrow & (\sigma, \sigma\theta) & \leftarrow \text{dual} \rightarrow & (\sigma\theta, \sigma) \end{array}$$

Previously, we used the action of σ on Φ_θ to determine the σ-split portion inside the θ-split torus. Similarly, we look at the action of $\sigma\theta$ on Φ_θ to find the $\sigma\theta$-split portion inside the θ-split torus. Let the maximal $\mathbb{R}$-split torus for (θ, σ) be A (as usual) and the maximal $\mathbb{R}$-split torus for $(\theta, \sigma\theta)$, S. So $S^-_{\sigma\theta}$ is maximal $\sigma\theta$-split and θ-split which is equivalent to S^+_σ which is a maximal in the fixed point group.

Definition 5. Let A and S be as above. The singular rank is the difference in rank of the (σ, θ)-stable maximal $(\sigma, \mathbb{R})$-split torus and the (σ, θ)-stable maximal $\mathbb{R}$-split, σ-fixed torus. The singular rank is calculated as follows:

$$\text{singular rank} = \dim(A^-_\sigma) + \dim(S^-_{\sigma\theta}) - \dim(A).$$

The singular rank helps to determine the maximal singular involution. From there we determine the appropriate structure of the remaining classes between the maximal σ-split and the maximal σ-fixed ($\theta\sigma$-split). It has been shown that under certain conditions of H_k (namely H_k pseudo-connected), one uses representatives of the same conjugacy classes in the H_k or $G_{\sigma\theta}$ (the fixed point group of $\sigma\theta$).

Proposition 14 ([6, Proposition 9.24]). *Let w_1 and w_2 be (σ, θ)-singular involutions and let H_k be pseudo-connected. Then the following are equivalent.*

1. *w_1 and w_2 are conjugate under $W(A, H_k^+)$.*
2. *w_1 and w_2 are conjugate under $W(A_\sigma^-, H_k^+)$.*
3. *w_1 and w_2 are conjugate under $W(A, G_{\sigma\theta})$.*
4. *w_1 and w_2 are conjugate under $W(A_\sigma^-, G_{\theta\sigma})$.*

In some cases, the number of H_k-conjugacy classes is determined quickly because the singular rank is maximal or 0. The final caveat is that while the structure from the $A^{(\theta,\sigma)}/H$ conjugacy classes is useful here when one considers the $H_\mathbb{R}$-conjugacy classes in $W(A)$, then involutions that were previously conjugate split as demonstrated in the following example.

Example. Consider the case for $\ell = 7, p = 2, q = 4, i = 1$ from he general case in Table 2.

- $\Phi_\theta = \Phi(A) = BC_2$ and $\Phi_{\theta,\sigma} = \Phi(A, A_\sigma^-) = BC_2$.
- The rank of the maximal σ-split torus is 2 and the rank of the maximal σ-fixed torus is also 2, but the rank of the maximal $\mathbb{R}$-split (i.e., θ-split) torus is also 2. Then the "top" the maximal $\mathbb{R}$-split torus is a σ-split torus and the "bottom" the torus is a σ-fixed torus.
- Consider the tori that are standard to A where $\dim((A_i)_\sigma^-)$= 2,1, and 0.

Through direction computation on the six roots in $\Phi(A, A_\sigma^-)^+$ ($e_1 \pm e_2$, e_1, $2e_1$, e_2, $2e_2$) the two (θ, σ)-singular roots can be determined. These roots are the unique short roots, usually denoted e_1 and e_2.

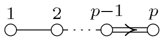
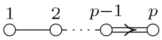
In $W(A)$, roots of type A_1 are conjugate. So the conjugacy classes of (σ, θ)-singular roots in $W(A)$ are the blackened dots in the diagram in Figure 3. This classifies the quasi $\mathbb{R}$-split tori that are H-conjugate to a maximal $\mathbb{R}$-split torus.

FIGURE 3. $A_0^{(\theta,\sigma)}/H$ & $A_\mathbb{R}^{(\theta,\sigma)}/H_\mathbb{R}$

There is one conjugacy class at each level. So all tori $A_i \in A_0^{(\sigma,\theta)}$ such that $\dim((A_i)_\sigma^-) = 2$ are conjugate. Similarly those with dimension 1 and 0. However, if we consider the conjugacy classes of these singular involutions in $W(A, H_\mathbb{R}^+) = BC_1 + BC_1$, then e_1 and e_2 are both type A_1 but no longer conjugate. So the one-dimensional level will split and there will two $H_\mathbb{R}^+$-conjugacy classes of tori where the rank of the σ-split portion is 1. It should be noted that these calculations are done in the associated Lie algebra and lifted to the group.

My current research is to complete the classification of the $H_\mathbb{R}^+$-conjugacy classes thus completing the classification of orbits of parabolic $\mathbb{R}$-subgroups on the symmetric space $G_\mathbb{R}/H_\mathbb{R}$, $P_\mathbb{R} \setminus G_\mathbb{R}/H_\mathbb{R}$.

Type (θ, σ)	Type $(\theta, \sigma\theta)$
$A_\ell^{p,q}(\mathrm{III_a}, \mathrm{III_a}, \epsilon_\mathrm{i})$ $p < q - p + 2i \le \frac{1}{2}(\ell+1)$ $1 \le p < q \le \frac{1}{2}(\ell+1)$ $(0 \le i \le p)$	$A_\ell^{p,q-p+2i}(\mathrm{III_a}, \mathrm{III_a}, \epsilon_\mathrm{p-i})$

rank $\Phi_{\sigma,\theta}$	$\sigma \vert \Phi_\theta$	$\Phi_{\sigma,\theta} \cap \Phi_\theta$	rank $\Phi_{\sigma\theta,\theta}$	$\sigma\theta \vert \Phi_\theta$	$\Phi_{\sigma\theta} \cap \Phi_\theta$
p		BC_p	p		BC_p

max.involution $\Phi_{\sigma,\theta} \cap \Phi_\theta$	Type Φ_θ	W-conjugacy classes	max. involution $\Phi_{\sigma\theta,\theta} \cap \Phi_\theta$	singular rank
BC_p	BC_p	$B(p)$	BC_p	p

$\mathfrak{g}_{\sigma\theta Int(\epsilon_i)}(\mathbb{R})$	$\Phi(\mathfrak{t}_1) + \Phi(\mathfrak{t}_2)$
$su(\ell+1-q-i, p-i) + su(q-p+i, i) + so(2)$	$BC_i + BC_{p-i}$

TABLE 2

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Pencils of Conics as a Classification Code

Vladimir Dragović

Abstract. We collect several subjects of the modern Mathematical Physics like integrable quad-graphs, discriminantly separable polynomials, the Petrov classification, the algebro-geometric approach to the Yang-Baxter equation and quadrirational maps since they all lead to the same geometric background. The geometry is related to pencils of conics, and the classification code follows the types of pencils.

Mathematics Subject Classification (2010). 14H70, 37K20, 37K60 (82A69, 83C20).

Keywords. Pencil of conics, Petrov classification, integrable quad-graphs, discrminantly separable polynomials, Yang-Baxter equation, quadrirational maps.

1. Pencils of conics

Given two conics in the plane, the set of all conics sharing the same intersection with the two, forms a pencil of conics. We will denote general pencils of conics having four simple common points of intersection as $(1, 1, 1, 1)$, or of type [A]. The case with two simple points of intersection and one double with a common tangent at that point is denoted $(1, 1, 2)$ or [B]. The case with two double points of intersection and with a common tangent in each of them is $(2, 2)$, or [C]. The case $(1, 3)$, denoted also as [D] is defined by one simple and one triple point of intersection. Finally (4), the case of one quadruple point is denoted as [E]. The following Figures 1–5 illustrate these possible configurations of pencils.

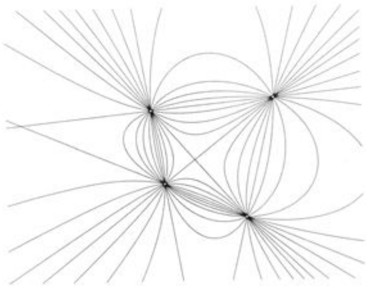

FIGURE 1. Pencil of type A

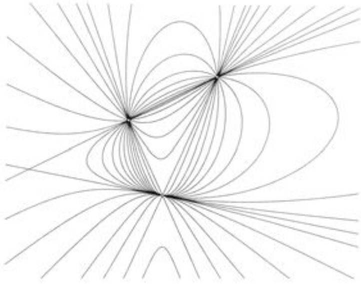

FIGURE 2. Pencil of type B

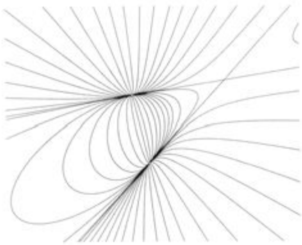

FIGURE 3. Type C

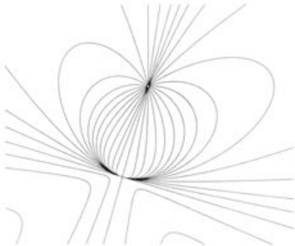

FIGURE 4. Type D

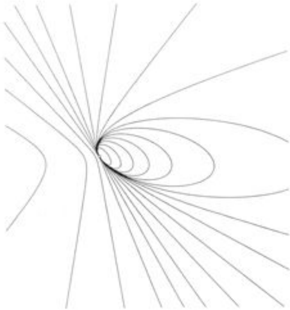

FIGURE 5. Type E

The transition from a more general pencil to a more special one is represented by the diagram, which is usually associated with Penrose:

$$\begin{array}{ccccc} A & & & & \\ \downarrow & \searrow & & & \\ B & \longrightarrow & C & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ D & \longrightarrow & E & \longrightarrow & O \end{array} \tag{1}$$

We will need a classical notion of the Darboux coordinates in a projective plane. We fix a conic C in the plane, with a rational parametrization. For a given point P in the plane, there are two tangents from P to the conic C. Let the two values of the rational parameter of the two points of tangency of the tangent lines with the conic C be (x_1, x_2). Then, the pair (x_1, x_2) gives *the Darboux coordinates* of the point P associated with the parametrized conic C.

2. Petrov classification

We will start with historically the first of the stories. The Petrov 1954 classification describes the algebraic symmetries of the Weyl tensor at a point in a Lorentzian manifold (see [1], [2]). It is well known due to its applications to the theory of relativity, in the study of the exact solutions of the Einstein field equations.

The Weyl tensor, is a $(2,2)$-tensor, evaluated at some point, and it acts on the space of bivectors at that point as a linear operator:

$$W : Y^{\alpha\beta} \mapsto \frac{1}{2} W_{pq}^{\alpha\beta} Y^{pq}. \tag{2}$$

The equation

$$W_{pq}^{\alpha\beta} Y^{pq} = 2\lambda Y^{\alpha\beta}$$

defines the eigenvalues and the eigenbivectors. In the case of a space-time of dimension four, the space of antisymmetric bivectors at a point is of dimension six, and, due to the symmetries of the Weyl tensor, the eigenbivectors lie in a subspace of dimension four. Thus, the Weyl tensor at each point has at most four linearly independent eigenbivectors. The eigenbivectors of the Weyl tensor can occur with multiplicities, indicating a kind of algebraic symmetry of the tensor at the point. The multiplicities reflect the structure of zeros of a certain polynomial of degree four. The eigenbivectors are associated with null vectors in the original space-time, the principal null directions at point. According to the Petrov classification theorem, there are six possible types of algebraic symmetry, the six Petrov types:

[I] four simple principal null directions;
[II] two simple principal null directions and one double;
[D] two double principal null directions;
[III] one simple and one triple principal null direction;
[N] one quadruple principal null direction.
[O] the case where the Weyl tensor vanishes.

A relationship between the Petrov classification and the pencils of conics has been elaborated in [3]. It has been represented by a diagram of type (1) by Penrose, see [4], with the following correspondence

$$(A, B, C, D, E, 0) \to (\mathrm{I}, \mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}, 0).$$

3. Integrable quad-graphs

Let us denote by $\mathcal{P}_d^n$ the set of polynomials in d variables of degree at most n in each.

Recall that the basic building blocks of systems on quad-graphs from works of Adler, Bobenko, Suris [5] are the equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{3}$$

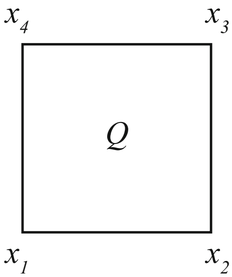

Figure 6. Quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

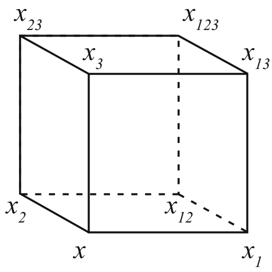

Figure 7. A 3D consistency.

where $Q \in \mathcal{P}_4^1$. Equations of type (3) are called *quad-equations.* The field variables x_i are assigned to four vertices of a quadrilateral as in Figure 6.

Following [5] we consider the idea of integrability as consistency, see Figure 7. We assign six quad-equations to the faces of coordinate cube. The system is said to be *3D-consistent* if three values for x_{123} obtained from equations on right, back and top faces coincide for arbitrary initial data x, x_1, x_2, x_3. Then, applying discriminant-like operators introduced in [5] $\delta_{x,y} : \mathcal{P}_4^1 \to \mathcal{P}_2^2, \quad \delta_x : \mathcal{P}_2^2 \to \mathcal{P}_1^4$ by formulae

$$h(z, w) := \delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad P(z) := \delta_w(h) = h_w^2 - 2h h_{ww}, \tag{4}$$

there is a descent from the faces to the edges and then to the vertices of the cube: from a polynomial $Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1$ to a biquadratic polynomial $h \in \mathcal{P}_2^2$ and further, to a polynomial $P \in \mathcal{P}_1^4$ of one variable of degree 4.

A biquadratic polynomial $h(x, y) \in \mathcal{P}_2^2$ is said to be *non degenerate* if no polynomial in its equivalence class with respect to fractional linear transformations is divisible by a factor of the form $x - c$ or $y - c$, with $c = \text{const}$. A multiaffine function $Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1$ is said to be of *type* Q if all four of its accompanying biquadratic polynomials h^{jk} are non degenerate. Otherwise, it is of *type* H. Previous notions were introduced in [5], where the classification list of mulitiaffine polynomials of type Q has been obtained, based on the structure of zeros of the associated nonzero polynomial P of degree four. There are five cases, [A], [B], [C], [D], [E]. For example, in the case $[B] = (1, 1, 2)$:

$$\begin{aligned} Q_B = (\alpha - \alpha^{-1})(x_1 x_2 + x_3 x_4) + (\beta - \beta^{-1})(x_1 x_4 + x_2 x_3) \\ - (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1 x_3 + x_2 x_4) \\ + \frac{\delta}{4}(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}) \end{aligned}$$

for $\delta \neq 0$. In the case $[C] = (2, 2)$ Q_C is obtained from Q_B with $\delta = 0$.

4. Discriminantly separable polynomials

The notion of discriminantly separable polynomials has been introduced in [6]. A family of such polynomials has been constructed there as pencil equations from the theory of conics $\mathcal{F}(w, x_1, x_2) = 0$, where w, x_1, x_2 are the pencil parameter and the Darboux coordinates respectively. The key algebraic property of the pencil equation, as quadratic equation in each of three variables w, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\mathcal{D}_w(\mathcal{F}) = P(x_1)P(x_2),\ \mathcal{D}_{x_1}(\mathcal{F}) = J(w)P(x_2)\,\mathcal{D}_{x_2}(\mathcal{F}) = P(x_1)J(w), \tag{5}$$

where J, P are polynomials of degree up to 4, and the elliptic curves $\Gamma_1 : y^2 = P(x)$, $\Gamma_2 : y^2 = J(s)$ are isomorphic (see Proposition 1 of [6]).

A classification of strongly discriminantly separable polynomials

$$\mathcal{F}(x_1, x_2, x_3) \in \mathcal{P}_3^2,$$

which are those satisfying the above relations 5 with $P = J$, has been performed modulo a gauge group of the following fractional-linear transformations $x_i \mapsto (ax_i + b)/(cx_i + d), \quad i = 1, 2, 3$ in [7], where more details can be found.

The classification of such polynomials, following [7], goes along the study of structure of zeros of a nonzero polynomial $P \in \mathcal{P}_1^4$. There are five cases: [A] with four simple zeros; [B] with a double zero and two simple zeros; [C] corresponds to polynomials with two double zeros; [D] is the case of one triple and one simple zero; finally, [E] is the case of one zero of degree four. The corresponding families of polynomials $\mathcal{F}_A$, $\mathcal{F}_B$, $\mathcal{F}_{C1}$, $\mathcal{F}_{C2}$, $\mathcal{F}_D$, $\mathcal{F}_{E1}$, $\mathcal{F}_{E2}$, $\mathcal{F}_{E3}$, $\mathcal{F}_{E4}$ are listed in Theorem 4 of [7]. Here, we are giving an example.

[B] $(1, 1, 2)$: two simple zeros and one double zero, for a canonical form of the polynomial $P(x) = x^2 - \epsilon^2$, the corresponding discriminantly separable polynomial is $\mathcal{F}_B = x_1x_2x_3 + (\epsilon/2)(x_1^2 + x_2^2 + x_3^2 - \epsilon^2)$.

The relationship between the discriminantly separable polynomials of degree two in each of three variables, and integrable quad-graphs of Adler, Bobenko and Suris has been established in [7]. The key point is the following formula, which defines an h, a biquadratic ingredient of quad-graph integrability, starting form a discriminantly separable polynomial $\mathcal{F}$: $\hat{h}(x_1, x_2, \alpha) = \mathcal{F}(x_1, x_2, \alpha)/\sqrt{P(\alpha)}$.

5. Quantum Yang-Baxter equation

The next subject is devoted to *the Yang–Baxter equation*

$$R^{12}(t_1 - t_2, h)R^{13}(t_1, h)R^{23}(t_2, h) = R^{23}(t_2, h)(R^{13}(t_1, h)R^{12}(t_1 - t_2, h). \tag{6}$$

Here t is so-called *spectral parameter* and h is *Planck constant.* Here we assume that $R(t, h)$ is a linear operator from $V \otimes V$ to $V \otimes V$ and $R^{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$ is an operator acting on the ith and jth components as $R(t, h)$ and as identity on

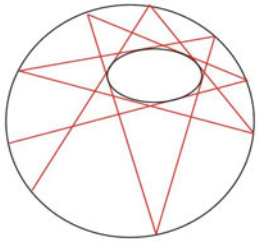

Figure 8. The Euler-Chasles correspondence

the third component. For example $R^{12}(t,h) = R \otimes Id$. In the first nontrivial case, matrix $R(t,h)$ is 4×4 and the space V is two-dimensional.

Krichever's approach is based on the *vacuum vector representation* of a 4×4 matrix L, understood as a 2×2 matrix with blocks of 2×2 matrices. In other words, $L = L^{i\alpha}_{j\beta}$ is a linear operator in the tensor product $C^2 \otimes C^2$. *The vacuum vectors* X, Y, U, V satisfy, by definition, the relation

$$LX \otimes U = hY \otimes V. \tag{7}$$

The vacuum vectors are parametrized by *the vacuum curve* Γ_L. In [8] Krichever proved that in the case of general position, the vacuum curve is elliptic, and rank one solutions are equivalent to the Baxter R-matrix. In [9], [10] the cases of rational vacuum curves have been studied.

The geometric background of the above algebro-geometric classification is connected with pencils of conics. It is based on the fact that the vacuum curve is a biquadratic, or the Euler-Chasles 2-2 correspondence (see [11]) of the form

$$E : ax^2y^2 + b(x^2y + xy^2) + c(x^2 + y^2) + 2dxy + e(x + y) + f = 0. \tag{8}$$

Using the Darboux coordinates, we visualize the Euler-Chasles correspondence (8) by Figure 8 and a relationship with pencils of conics becomes obvious. Thus, again, the classification follows the Penrose diagram (1) where the case [A] corresponds to the Baxter R-matrix, [B] to the Cherednik R-matrix, and [C] to the six-vertex R-matrix of Yang.

6. Quadrirational maps

The last section is devoted to quadrirational maps on $\mathbb{CP}^1$ which are introduced and classified in [12]. Following Adler, Bobenko and Suris, we consider a rational map $F : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and its graph as an algebraic variety $\Gamma_F \subset (\mathbb{CP}^1)^4$. Such a map is called *quadrirational* if for any fixed pair $(X, Y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ (modulo some closed subvariety of co-dimension at least one) the graph Γ_F intersects each of the sets $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \{X\} \times \{Y\}$, $\{X\} \times \{Y\} \times \mathbb{CP}^1 \times \mathbb{CP}^1$,

$\mathbb{CP}^1 \times \{Y\} \times \{X\} \times \mathbb{CP}^1$ exactly once. In that case Γ_F defines four rational maps $F, F^{-1}, \bar{F}, \bar{F}^{-1} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$. It has been proven in [12] that for a quadrirational map, its graph is defined by polynomial equations $f(x, y, u) = 0$ and $h(y, x, v) = 0$, where the degrees of f in x and of h in y are one or two. We will consider further only the case when both of the degrees are equal to two, denoted in [12] as $[2 : 2]$. Then, the following classification takes place:

Theorem (Adler, Bobenko, Suris 2004). *Any quadrirational map of type* $[2 : 2]$ *is, up to Möbius gauge transformations on variables, equivalent to one and only one of the five maps:*

[A] $\quad F_A : u = ayP, \quad v = bxP, \quad P = \dfrac{(1-b)x + b - a + (a-1)y}{b(1-a)x + (a-b)yx + a(b-1)y};$

[B] $\quad F_B : u = \dfrac{y}{a}P, \quad v = \dfrac{x}{b}P, \quad P = \dfrac{ax - by + b - a}{x - y};$

[C] $\quad F_C : u = \dfrac{y}{a}P, \quad v = \dfrac{x}{b}P, \quad P = \dfrac{ax - by}{x - y};$

[D] $\quad F_D : u = yP, \quad v = xP, \quad P = \dfrac{x - y + b - a}{x - y};$

[E] $\quad F_E : u = y + P, \quad v = x + P, \quad P = \dfrac{b - a}{x - y};$

where a, b are given constants.

The mappings F_A, F_B, F_C, F_D, F_E are related with pencils of conics of types A, B, C, D, E respectively, in the following way: given two conics C_1, C_2 of a pencil, with fixed rational parametrizations. For a pair of points $x \in C_1, y \in C_2, x \neq y$, the line they define intersects conics C_1 and C_2 in other two points u, v. Then, as it has been shown in [12], $F(x, y) = (u, v)$ is a quadrirational mapping, with the formula given above.

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Geodesic Mappings and Einstein Spaces

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Abstract. In this paper we study fundamental properties of geodesic mappings with respect to the smoothness class of metrics. We show that geodesic mappings preserve the smoothness class of metrics. We study geodesic mappings of Einstein spaces.

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Keywords. Geodesic mapping, smoothness class, Einstein space.

1. Introduction

First we study the general dependence of geodesic mappings of (pseudo-) Riemannian manifolds in dependence on the smoothness class of the metric. We present well-known facts, which were proved by Beltrami, Levi-Civita, Weyl, Sinyukov, etc., see [1–5]. In these results no details about the smoothness class of the metric were discussed. They were formulated "for sufficiently smooth" geometric objects.

In the last section we present proofs of some facts about geodesic mappings of Einstein spaces.

2. Geodesic mappings theory for $V_n \to \bar{V}_n$ of class C^1

Assume the (pseudo-) Riemannian manifolds $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ with metrics g and $\bar{g}$, and Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here V_n, $\bar{V}_n$ $\in C^1$, i.e., $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Definition 1. A diffeomorphism f: $V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

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Let there exist a geodesic mapping $f\colon V_n \to \bar{V}_n$. Since f is a diffeomorphism, we can assume the existence of local coordinate maps on M or $\bar{M}$, respectively, such that locally, $f\colon V_n \to \bar{V}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$. A manifold V_n admits a geodesic mapping onto $\bar{V}_n$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{1}$$

hold for any tangent fields X, Y and where ψ is a differential form. If $\psi \equiv 0$ than f is *affine* or *trivially geodesic.*

In a local form: $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i$, where $\Gamma^h_{ij}(\bar{\Gamma}^h_{ij})$ are the Christoffel symbols of V_n and $\bar{V}_n$, ψ_i are components of ψ and δ^h_i is the Kronecker delta. Equations (1) are equivalent to the following equations

$$\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi\bar{g}_{ik} \tag{2}$$

where "," denotes the covariant derivative in V_n. It is known that

$$\psi_i = \partial_i\Psi, \quad \Psi = \frac{1}{2(n+1)}\ln\left|\frac{\det\bar{g}}{\det g}\right|, \quad \partial_i = \partial/\partial x^i.$$

Sinyukov [5] proved that the Levi-Civita equations are equivalent to

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{3}$$

where

$$a_{ij} = \mathrm{e}^{2\Psi}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}; \quad \lambda_i = -\,\mathrm{e}^{2\Psi}\bar{g}^{\alpha\beta}g_{\beta i}\psi_\alpha.$$

From (3) follows $\lambda_i = \partial_i\lambda = \partial_i(\frac{1}{2}\,a_{\alpha\beta}g^{\alpha\beta})$. On the other hand [4, p. 63]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi}\tilde{g}_{ij}, \quad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde{g}}{\det g}\right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{4}$$

The above formulas are the criterion for geodesic mappings $V_n \to \bar{V}_n$ globally as well as locally.

3. Geodesic mappings theory for $V_n \to \bar{V}_n$ of class C^2

Let V_n and $\bar{V}_n \in C^2$, then for geodesic mappings $V_n \to \bar{V}_n$ the Riemann and the Ricci tensors transform in the following way

$$\text{(a)}\quad \bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik}; \quad \text{(b)}\quad \bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij}, \tag{5}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, and the Weyl tensor of projective curvature, which is defined in the following form $W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$, is invariant.

The integrability conditions of the Sinyukov equations (3) have the following form

$$a_{i\alpha}R^\alpha_{jkl} + a_{j\alpha}R^\alpha_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}. \tag{6}$$

After contraction with g^{jk} we get [5]

$$n\lambda_{i,l} = \mu g_{il} + a_{i\alpha}R^\alpha_l - a_{\alpha\beta}{R^\alpha}_{il}{}^\beta \tag{7}$$

where ${R^\alpha}_{ij}{}^\beta = g^{\beta k}{R^\alpha}_{ij\beta}$; $R^\alpha_j = g^{\alpha\beta}R_{\beta j}$ and $\mu = \lambda_{i,j}g^{ij}$.

4. Geodesic mappings between $V_n \in C^r$ $(r > 2)$ and $\bar{V}_n \in C^2$

Theorem 1. *If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.*

The proof of this theorem follows from the following lemmas.

Lemma 2. *Let $\lambda^h \in C^1$ be a vector field and ϱ a function.*
If $\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.

Proof. The condition $\partial_i\lambda^h - \varrho\,\delta^h_i \in C^1$ can be written in the following form

$$\partial_i\lambda^h - \varrho\delta^h_i = f^h_i(x), \tag{8}$$

where $f^h_i(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $h \neq i$ we integrate (8) with respect to dx^i:

$$\lambda^h = \Lambda^h + \int_{x^i_o}^{x^i} f^h_i(x^1,\dots,x^{i-1},t,x^{i+1},\dots,x^n)\,dt,$$

where Λ^h is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^h and the above integrals (see [6, p. 300]), also the derivatives $\partial_h\Lambda^h$ exist; in this proof we don't use the Einstein's summation convention. Then we can write (8) for $h = i$:

$$\varrho = -f^h_h + \partial_h\Lambda^h + \int_{x^i_o}^{x^i} \partial_h f^h_i(x^1,\dots,x^{i-1},t,x^{i+1},\dots,x^n)\,dt. \tag{9}$$

Because the derivative with respect to x^i of the right-hand side of (9) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i\varrho = \partial_h f^h_i - \partial_i f^h_h$, therefore $\varrho \in C^1$ and from (8) follows $\lambda^h \in C^2$. □

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i\lambda^h - \varrho\delta^h_i \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 3. *If $V_n \in C^3$ admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^3$.*

Proof. In this case the Sinyukov's equations (3) and (7) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. Then by a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^h_{ijk}, R^h{}_{ij}{}^k, R_{ij}, R^h_i \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (7) in the form (8), where $\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1$, $\varrho = \mu/n$ and $nf^h_i = -\lambda^\alpha\Gamma^h_{\alpha i} + g^{h\gamma}a_{\alpha\gamma}R^\alpha_i - a_{\alpha\beta}R^h{}_{\alpha\beta i} \in C^1$.

From Lemma 2 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (3) twice we demonstrate that $a_{ij} \in C^3$. From this and formula (4) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$. □

Further we notice that for geodesic mappings between V_n and $\bar{V}_n$ of class C^3 holds the third set of Sinyukov equations:

$$(n-1)\mu_{,i} = 2(n+1)\lambda_\alpha R^\alpha_k + a_{\alpha\beta}(2R^\alpha{}_{k,}{}^\beta - R^{\alpha\beta}{}_{,k}). \tag{10}$$

If $V_n \in C^r$ and $\bar{V}_n \in C^2$, then by Lemma 3, $\bar{V}_n \in C^3$ and (10) hold. Because Sinyukov's system (3), (7) and (10) is closed, we can differentiate equations (3) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{V}_n \in C^r)$.

Remark 4. Because for holomorphically projective mappings of Kähler (and also hyperbolic and parabolic Kähler) spaces hold equations analogical to (3) and (7), see [3, 5, 7], from Lemma 2 follows an analog to Theorem 1 for these mappings.

5. On geodesic mappings of Einstein spaces

Geodesic mappings of Einstein spaces were studied by many authors starting with A.Z. Petrov (see [8]). Einstein spaces V_n are characterized by the condition Ric $=$ const $\cdot\, g$, so $V_n \in C^2$ would be sufficient. But many properties of Einstein spaces occur when $V_n \in C^3$ and $n > 3$. An Einstein space V_3 is a space of constant curvature.

We continue with geodesic mappings of Einstein spaces $V_n \in C^3$. On basis of Theorem 1 it is natural to suppose that $\bar{V}_n \in C^3$. In 1978 (see PhD. thesis [9] and [10]) Mikeš proved that under these conditions the following theorem holds:

Theorem 5. *If the Einstein space V_n admits a nontrivial geodesic mapping onto a* (*pseudo-*) *Riemannian space $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Proof. Let the Einstein space $V_n \in C^3$ (for which $R_{ij} = -K\,(n-1)\,g_{ij}$) admit a nontrivial geodesic mapping onto $\bar{V}_n \in C^2$. Then the Sinyukov equations (3) hold; their integrability conditions have the form given in (6). Taking (3) into account, we differentiate (6) with respect to x^m, contract the result with g^{lm}, and then we alternate with respect to i, k. By (8), we get $\lambda_\alpha R^\lambda_{ijk} = g_{ij}\xi_k - g_{ik}\xi_j$, where ξ_i is some vector. Contracting the latter with g^{ij} and using (8) we see that $\xi_i = K\lambda_i$, that is, the formula reads $\lambda_\alpha R^\lambda_{ijk} = K(g_{ij}\lambda_k - g_{ik}\lambda_j)$.

We contract (6) with λ^l. Considering the last formula, we get

$$g_{ki}\Lambda_{j\alpha}\lambda^\alpha + g_{kj}\Lambda_{i\alpha}\lambda^\alpha - \lambda_i\Lambda_{jk} - \lambda_j\Lambda_{ik} = 0, \tag{11}$$

where $\Lambda_{ij} = \lambda_{i,j} - K a_{ij}$. It is easy to show that $\lambda^a\Lambda_{\alpha i} = \mu\lambda_i$, where μ is a function. Since $\lambda_i \neq 0$, we find from (11) that

$$\lambda_{i,j} = \mu\, g_{ij} + K\, a_{ij}. \tag{12}$$

Differentiating (12) and considering (3), (7), it is easy to obtain the following equation:

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i\psi_j = \bar{K}\, g_{ij} - K\bar{g}_{ij}, \tag{13}$$

where $\bar{K}$ is a function. Then from (5), by virtue of the last relation, and considering $R_{ij} = -K\,(n-1)\,g_{ij}$, we get that $\bar{R}_{ij} = (n-1)\bar{K}\,\bar{g}_{ij}$. Hence $\bar{V}_n$ is an Einstein space. The theorem is proved. □

Theorem 5 was proved "locally" but it is easy to show that when the domain of validity of equations (13) border with a domain where $\psi_i \equiv 0$, then in this domain $\psi_i \equiv 0$. Assume a point x_0 on the border between these domains, then

$\psi_i(x_0) = 0$ and $\psi_{ij} = 0$. Indeed **a)** If $K \neq 0$ or $\bar{K} \neq 0$ then $\bar{g}_{ij}(x_0) = \bar{K}/K\ g_{ij}(x_0)$. From these properties follows that the system of equations (2) and (13) has a unique solution $\bar{g}_{ij} = \bar{K}/K\ g_{ij}$ and $\psi_i = 0$. **b)** If $K = \bar{K} = 0$ then equations (13): $\psi_{i,j} = \psi_i\psi_j$ have a unique solution for $\psi_i(x_0) = 0$: $\psi_i = 0$.

This Theorem was used for geodesic mappings of four-dimensional Einstein spaces (Mikeš, Kiosak [11]) and to find metrics of Einstein spaces that admit geodesic mappings (Formella, Mikeš [12]).

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Racah Operators

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Abstract. We introduce the notion of Racah operators (as a generalization of Racah coefficients) for representations of groups. We then describe these operators explicitly for the motion group of the three-dimensional space. The connection with the geometry of spatial polygons (hinge polygons) is explained.

Mathematics Subject Classification (2010). Primary 20C99; Secondary 20C33.

Keywords. Racah operators, group representations, geometry of spatial polygons.

1. Racah operators for a product of three representations

Let let G be a locally compact group with the "nice" dual space $\hat{G}$ ("nice" means, in particular, that any unitary representation of G admits a unique decomposition in irreducible representations) and $\{H_c\}, c \in \hat{G}$, – a family of Hilbert spaces carrying representations T_c of equivalence class $c \in \hat{G}$ (these families also are assumed to be "nice"). Let for any pair $a_1, a_2 \in \hat{G}$ be given a fixed decomposition of the product $T_{a_1} \otimes T_{a_2}$ into irreducible representations. More exactly we have a unitary isomorphisms

$$H_{a_1} \otimes H_{a_2} \simeq \int_{\hat{G}} (H_c \otimes V_c^{a_1,a_2}) d\tau^{a_1,a_2}(c).$$

Here $\{V_c^{a_1,a_2}, c \in \hat{G}\}$ denotes a family of Hilbert spaces (this family is "nice" of course as well as the mapping $(a_1, a_2) \mapsto \tau^{a_1,a_2}$). The group G acts in $H_c \otimes V_c^{a_1,a_2}$ as $T_c \otimes I_c$ where I_c denotes the trivial representation in $V_c^{a_1,a_2}$.

Then take a product $T_{a_1} \otimes T_{a_2} \otimes T_{a_3}$ and decompose it following the usual "Racah strategy": first we decompose $T_{a_1} \otimes T_{a_2}$ as indicated and then decompose products $T_c \otimes T_{a_3}$ for any $c \in \hat{G}$. We obtain a measure $d\tau(c,l) = d\tau^{c,a_3}(l) \cdot d\tau^{a_1,a_2}(c)$ on $\hat{G} \times \hat{G}$ and an isomorphism

$$H_{a_1} \otimes H_{a_2} \otimes H_{a_3} \simeq \int_{\hat{G}\times\hat{G}} (H_l \otimes V_c^{a_1,a_2} \otimes V_l^{c,a_3}) d\tau(l,c).$$

We can rewrite this as

$$H_{a_1} \otimes H_{a_2} \otimes H_{a_3} \simeq \int\limits_{\hat{G}} (H_l \otimes M_l) d\theta(l), \tag{1}$$

where

$$M_l = \int\limits_{\hat{G}} (V_c^{a_1,a_2} \otimes V_l^{c,a_3}) d\mu^l(c).$$

We now repeat this construction decomposing first $T_{a_2} \otimes T_{a_3}$ in $\{T_c\}$ and then decomposing the products $T_{a_1} \otimes T_c$ for any c. This gives another isomorphism

$$H_{a_1} \otimes H_{a_2} \otimes H_{a_3} \simeq \int\limits_{\hat{G}} (H_l \otimes N_l) d\sigma(l), \tag{2}$$

where

$$N_l = \int\limits_{\hat{G}} (V_l^{a_1,c} \otimes V_c^{a_2,a_3}) d\nu^l(c).$$

Equations (1) and (2) lead to a unitary isomorphism

$$\int\limits_{\hat{G}} (H_l \otimes M_l) d\theta(l) \simeq \int\limits_{\hat{G}} (H_l \otimes N_l) d\sigma(l), \tag{3}$$

compatible with the action of G. The measures θ and σ are equivalent and we can take $\theta = \sigma$. So the isomorphism (3) is determined by a family of unitary isomorphisms $R_l : M_l \to N_l$ or

$$R_l : \int\limits_{\hat{G}} (V_c^{a_1,a_2} \otimes V_l^{c,a_3}) d\mu^l(c) \to \int\limits_{\hat{G}} (V_l^{a_1,c} \otimes V_c^{a_2,a_3}) d\nu^l(c). \tag{4}$$

These are (by definition) the Racah operators.

In the most transparent case when all the products $T_{a_1} \otimes T_{a_2}$ have a simple spectra (so all the $V_c^{a_1,a_2}$ are one-dimensional) the Racah operators are

$$R_l : L_2(\hat{G}, d\mu^l(c)) \to L_2(\hat{G}, d\nu^l(c))$$

with some measures μ^l, ν^l on $\hat{G}$.

2. Racah operators for a product of many representations

As we can see the Racah operators arise if one writes the product $T_1 \otimes T_2 \otimes T_3$ first as $(T_1 \otimes T_2) \otimes T_3$ and then as $T_1 \otimes (T_2 \otimes T_3)$ and step by step decomposes the products of two representations contained between two brackets (opening and closing). Consider now a product $T_{a_1} \otimes T_{a_2} \otimes \cdots \otimes T_{a_n}, n \geq 3$. It also can be thought of as a result of step by step multiplication of two representations obtained by using a system of brackets (opening and closing); if for example $n = 4$ then the product $T_1 \otimes T_2 \otimes T_3 \otimes T_4$ can be written as $(T_1 \otimes T_2)(\otimes T_3 \otimes T_4)$, $((T_1 \otimes T_2)) \otimes T_3) \otimes T_4$ and so on. Any system of brackets gives a decomposition of $T_{a_1} \otimes T_{a_2} \otimes \cdots \otimes T_{a_n}, n \geq 3$ in irreducible representations. Given two different system of brackets we naturally come to a set of unitary operators relating these two decompositions (as we had above for $(T_1 \otimes T_2) \otimes T_3$ and $T_1 \otimes (T_2 \otimes T_3)$); these are the desired Racah operators.

3. The case of the motion group; construction for Racah operators

Let G be the group of motions of $\mathbf{R}^3$. Its elements are $g = (u, a)$, where $u \in SO(3), a \in \mathbf{R}^3$; (so g acts on $\mathbf{R}^3$ as $x \mapsto x \cdot u + a$).The subgroup $SO(3)$ acts on the sphere $S^2 = \{x \in \mathbf{R}^3 : (x, x) = 1\}$ by $x \cdot u$. The representation of G (which is not trivial on $\mathbf{R}^3 \subset G$,) is determined by a pair (m, l), where $m \in \mathbf{Z}$ and $l > 0$; denote it by $T^{m,l}$. It acts in $L_2(S^2)$; the subgroup $\mathbf{R}^3$ acts by multiplications $x \mapsto \exp(\imath l(a, x))$ and on the subgroup $SO(3)$ the representation is induced by the character $z \mapsto z^m$ of the subgroup $T^1 = \{z \in \mathbf{C} : |z| = 1\}$.

The product $T^{m_1,l_1} \otimes T^{m_2,l_2}$ is decomposed as

$$T^{m_1,l_1} \otimes T^{m_2,l_2} \simeq \int\limits_{\mathbf{Z}} \int\limits_{\Delta} T^{r,s} d(r, s).$$

Here $\Delta = (|l_1 - l_2|, l_1 + l_2)$ and integration on $\mathbf{Z}$ means summing.

In order to decompose $T^{m_1,l_1} \otimes T^{m_2,l_2} \otimes T^{m_3,l_3}$ we need the domain D_1 in the (s, l) -plane given by $|l_1 - l_2| \leq s \leq l_1 + l_2, |s - l_3| \leq l \leq s + l_3$. Let $h_0 = \min\{l : (s, l) \in D_1\}, h_1 = \max\{l : (s, l) \in D_1\}$. Then $h_1 = l_1 + l_2 + l_3$. Take the interval $\Delta = (h_0, h_1)$ and for any point $l \in \Delta$ take the interval $\omega_1(l) = \{s : (s, l) \in D_1\}$. Then the desired decomposition is

$$T^{m_1,l_1} \otimes T^{m_2,l_2} \otimes T^{m_3,l_3} \simeq \int\limits_{\mathbf{Z} \times \Delta} T^{m,l} \otimes I_1^{m,l} d(m, l), \tag{5}$$

where $I_1^{m,l}$ denotes the trivial representation in $L_2(\mathbf{Z} \times \omega_1(l))$.

Similarly, decomposing first $T^{m_2,l_2} \otimes T^{m_3,l_3}$ and then decomposing

$$T^{m_1,l_1} \otimes T^{r,s}$$

for all (r, s) we come to another decomposition

$$T^{m_1,l_1} \otimes T^{m_2,l_2} \otimes T^{m_3,l_3} \simeq \int\limits_{\mathbf{Z} \times \Delta} T^{m,l} \otimes I_2^{m,l} d(m, l). \tag{6}$$

Instead of the previous domain D_1 we take in the formula (6) the domain D_2 given by $|l_2 - l_3| \leq s \leq l_2 + l_3, |s - l_1| \leq l \leq l_1 + s$. $I_2^{m,l}$ denotes the trivial representation in $L_2(\mathbf{Z} \times \omega_2(l))$, where $\omega_2(l) = \{s : (s, l) \in D_2\}$.

The decompositions (5) and (6) give the following isomorphism:

$$\int\limits_{\mathbf{Z} \times \Delta} T^{m,l} \otimes I_1^{m,l} d(m, l) \simeq \int\limits_{\mathbf{Z} \times \Delta} T^{m,l} \otimes I_2^{m,l} d(m, l). \tag{7}$$

So the Racah operators are $R(m, l) : L_2(\mathbf{Z} \times \omega_1(l)) \to L_2(\mathbf{Z} \times \omega_2(l))$.

Now we pass to a more convenient realization of these operators. To do so replace any function $(r,s) \mapsto f(r,s)$ taken from the Hilbert space $L_2(\mathbf{Z}\times\omega_k(l)), k = 1,2$, by the Fourier series

$$(\theta,s) \mapsto f^\circ(\theta,s) = \sum\nolimits_r f(r,s)\theta^{-r}, (\theta,s) \in T^1 \times \omega_k(l), k = 1,2.$$

So the Hilbert spaces $L_2(\mathbf{Z} \times \omega_k(l))$ are replaced by $L_2(T^1 \times \omega_k(l)), k = 1,2$, and the Racah operators $R(l,\lambda)$ act as

$$R^\circ(m,l) : L_2(T^1 \times \omega_1(l)) \to L_2(T^1 \times \omega_2(l)). \tag{8}$$

We retain the name Racah operators for these actions and in next sections we will describe them.

4. Hinge transformation and explicit form for Racah operators for three representations

Fix positive numbers $a_i, 1 \le i \le 4$ such that any of them is less than the sum of others. Let $\mathcal{M}$ denote the set of polygons (in general position) $(A_1 \cdots A_4)$ with $|A_i, A_{i+1}| = a_i, 1 \le i \le 3, |A_1, A_4| = a_4$. These polygons are considered up to motions of the space $\mathbf{R}^3$. They are the *hinge* polygons (by definition). $\mathcal{M}$ is a smooth manifold (after removing the singular points). It can be parametrized in three ways: first we can take as parameters the values $|A_1, A_3|, |A_2, A_4|$, second – the values $|A_1, A_3|, \theta_{13}$ where θ_{13} is the angle between the planes determined by (A_1, A_2, A_3) and (A_1, A_4, A_3) and third – the values $|A_2, A_4|, \theta_{24}$ determined similarly. For any of these three parametrization introduce a 2-form by

$$(24V)^{-1} d|A_1, A_3|^2 \wedge d|A_2, A_4|^2, d|A_1, A_3| \wedge d\theta_{13}, d|A_2, A_4| \wedge d\theta_{24};$$

here V denotes the volume of the tetrahedron – the convex hull of our polygon.

It turns out that these three 2-forms coincide. Thus they lead to the same 2-form ω^2 on $\mathcal{M}$ – the surface form. It follows that the mapping

$$(|A_1, A_3|, \theta_{13}) \mapsto (|A_2, A_4|, \theta_{24})$$

preserves the two-dimensional Lebesque measure.

Return now to Racah operators. Let $\tau_{jk} = \exp(\imath\theta_{jk})$.

Theorem. *The Racah operator* (8) *is*

$$(R^\circ(m,l)f)(|A_2A_4|, \tau_{24}) = (\tau_{12})^{m_1}(-\tau_{23})^{m_2}(\tau_{34})^{m_3}(\tau_{14})^m f(|A_1A_3|, -\tau_{13}),$$
$$f \in L_2(T^1 \times \omega_1(l)).$$

5. Hinge transformation and explicit form for Racah operators for many representations

Consider a tensor product $T^{m_1,l_1} \otimes T^{m_2,l_2} \cdots \otimes T^{m_n,l_n}, n > 3$. As was explained above to follow the Racah strategy we must arrange the pairs of opening and closing brackets $(\ldots)$ between these T^{m_i,l_i} and then step by step decompose representations standing between two brackets into irreducible ones. Taking two systems of brackets we come to Racah operator which relates the corresponding decomposition. In order to realize this operator we need a generalization of the Hinge construction described above. More precisely consider a manifold $\mathcal{M}_n$ of n-polygons with fixed lengths $|A_i, A_{i+1}| = a_i$, $i = 1, \ldots, n-1$, $|A_1, A_n| = a_n$. Consider these polygons as images of the fixed convex polygon P_n^0 on the plane embedded into $\mathbf{R}^3$ isometrically on each A_iA_{i+1} and on A_1A_n. Triangulate P_n^0 by using diagonals and transport these diagonals to our polygon. So we come to parametrization of $\mathcal{M}_n$ by pairs $|A_i, A_{i+1}|, \theta_{i,i+1}$ and $|A_1, A_n|, \theta_{1,n}$ as we did above for $n = 4$. This leads to the volume-form on $\mathcal{M}_n$ defined as a product of forms $d|A_i, A_{i+1}| \wedge d\theta_{i,i+1}$ and $d|A_1, A_n| \wedge d\theta_{1,n}$. Two systems of parameters (which correspond to different triangulations of P_n^0) are related by measure preserving transformation (Hinge transformation).

The crucial fact is that the two constructions described here (the first for representations and the second one for polygons) are closely related. Moreover the Racah operators are realized as unitary operators in Hilbert spaces of L_2 type given by hinge transformation of polygons (and by multiplication operator).

The symplectic 2-form on $\mathcal{M}_n$ was discovered by A. Klyachko [1]. This also leads to a volume form: it turns out however that this volume form is not the same one, which we have constructed in Sections 4 and 5.

The general consideration of Racah operators is contained in [2] where also the case of the group $G = PSL(2, \mathbf{C})$ is described.

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q-discord for Generalized Entropy Functions

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Abstract. The aim of this article is to discuss how to define quantum correlations in composite systems. Based on the notion of a quantum discord we generalize it using other entropy functions than von Neumann entropy.

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Keywords. Quantum correlations, quantum discord, entropy, Tsallis entropy.

1. Introduction

One of the most intriguing topics in recent investigations of quantum information theory is the problem of quantifying correlations in composite quantum systems [1–4]. Especially interesting is the problem of establishing correlations which has pure quantum origin [5]. One of a possible measure of such correlations is a quantum discord introduced in [6, 7] and recently investigated in many aspects [8–20].

In what follows we will consider a bipartite system consisting of parts A and B described by finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. As a consequence, the Hilbert space of the total system is a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ and any state of the system is represented by a Hermitian, non negative semi-definite density matrix ρ_{AB} with $\mathrm{Tr}\rho_{AB} = 1$.

It is a common agreement that the most suitable quantity to measure correlations between subsystems is the *mutual information* (mutual entropy). This quantity is usually defined in terms of von Neumann entropy

$$S(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB} \log \rho_{AB})$$

as

$$I(A:B) = S(A) + S(B) - S(AB)\,, \tag{1}$$

where $S(A) = S(\rho_A)$, $S(B) = S(\rho_B)$, $S(AB) = S(\rho_{AB})$, $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ being reduced density matrices. Due to the subadditivity property (SA) of the von Neumann entropy [1, 21]

$$S(AB) \leq S(A) + S(B) \tag{2}$$

mutual information is non negative

$$I(A:B) \geq 0\,.$$

Clearly, $I(A:B)$ contains information about all correlations present in the state ρ_{AB} and one can put forward an equation displaying, which correlations are of classical origin and which are pure quantum. One can decompose (in a non unique way) the total correlations $I(A:B)$ as

$$I(A:B) \;=\; \underset{\text{quantum}}{D(A:B)} \;+\; \underset{\text{classical}}{C(A:B)}\,.$$

The classical part $C(A:B)$, as proposed in [6], can be determined in the following way: consider the set $\{\Pi_k\}$ of one-dimensional projectors fulfilling $\sum_k \Pi_k = \quad$, acting on the subsystem B and corresponding to some measurement procedure. The post-measurement states constitute an ensemble $\{p_k, \rho_k\}$, where

$$\rho_k \;=\; \frac{1}{p_k}(\quad \otimes \Pi_k)\rho_{AB}(\quad \otimes \Pi_k)^\dagger\,, \qquad k = 0, 1, \ldots\,,$$

and $p_k = \mathrm{Tr}(\quad \otimes \Pi_k)\rho_{AB}$, which can be used to define the *conditional entropy with respect to the measurement*

$$S(A|\{\Pi_k\}) \;=\; \sum_k p_k S(\rho_k)\,.$$

Finally, the so-called *Holevo quantity with respect to the measurement*

$$\chi(A|\{\Pi_k\}) \;=\; S(A) - S(A|\{\Pi_k\})$$

quantifies the amount of Π-type information contained in A and corresponds to some classical correlations between A and B provided by the measurement. Hence, *quantum correlations with respect to the measurement* are contained in the quantity

$$D(A:B|\{\Pi_k\}) = I(A:B) - \chi(A|\{\Pi_k\}), \tag{3}$$

which is called the *quantum discord with respect to* $\{\Pi_k\}$.

Now, classical correlations correspond to the maximal value of Holevo quantities obtained for different measurement procedures, i.e.,

$$C(A:B) = \sup_{\{\Pi_k\}} \chi(A|\{\Pi_k\}).$$

The strong subadditivity (SSA) of von Neumann entropy (see [1, 21])

$$S(ABC) + S(B) \leq S(AB) + S(BC)$$

results in the following pattern of implications

$$\text{SSA} \quad \Rightarrow \quad D(A:B|\Pi) \geq 0 \quad \Rightarrow \quad \text{SA}. \tag{4}$$

As a consequence of (4) the *quantum discord* is non negative [6, 7], i.e.,

$$D(A:B) \;=\; I(A:B) - C(A:B) \;\geq\; 0.$$

2. q-discord

The following question gives some motivations in our further investigations: can one generalize the notion of quantum discord to more general entropy functions? Going in this direction we consider the two-parameter family of entropy functions [22]

$$H_{q,s}(\rho) \;=\; \frac{1}{s(1-q)}\Big[(\mathrm{Tr}\rho^q)^s - 1\Big], \quad q,s>0,$$

which covers most of known entropies such as

- Renyi entropy for $s \to 0$,
- Tsallis entropy for $s = 1$,
- von Neumann entropy for $s = 1$ and $q \to 1$.

All the entropy functions $H_{q,s}$ are non negative, concave and, if ρ_{AB} is pure then $H_{q,s}(\rho_A) = H_{q,s}(\rho_B)$, but $H_{q,s}$ are no longer additive with respect to the tensor product, i.e.,

$$H_{q,s}(\rho_1 \otimes \rho_2) = H_{q,s}(\rho_1) + H_{q,s}(\rho_2) + s(1-q)H_{q,s}(\rho_1)H_{q,s}(\rho_2),$$

hence subadditivity (SA) fails in general, i.e., for arbitrary q, s. Note however that for Tsallis entropy $T_q \equiv H_{q,1}$ with $q > 1$ one obtains [23–26]

$$T_q(\rho_1 \otimes \rho_2) = T_q(\rho_1) + T_q(\rho_2) + (1-q)T_q(\rho_1)T_q(\rho_2) \le T_q(\rho_1) + T_q(\rho_2)$$

and moreover [27]

$$\text{for } q>1 \quad T_q(\rho_{AB}) \;\le\; T_q(\rho_A) + T_q(\rho_B), \tag{5}$$

hence SA holds.

Motivated by the validity of the property (5) for von Neumann entropies, in analogy to (3), we introduce the notion of a q-*discord with respect to the measurement* $\{\Pi_k\}$ as

$$D_q(A:B|\{\Pi_k\}) = I_q(A:B) - \chi_q(A|\{\Pi_k\})$$

where

$$I_q(A:B) = T_q(A) + T_q(B) - T_q(AB) \tag{6}$$

represents q-deformed total correlations and

$$\chi_q(A|\{\Pi_k\}) = T_q(A) - T_q(A|\{\Pi_k\})$$

is d-deformed Holevo quantity. Hence the q-discord is defined as

$$D_q(A:B) = I_q(A:B) - C_q(A:B),$$

where q-deformed classical correlations reads

$$C_q(A:B) = \sup\nolimits_{\{\Pi_k\}} \chi_q(A|\{\Pi_k\}).$$

Recall that T_q is SA for $q > 1$ but unfortunately fails to be SSA, in general, so we cannot use (4) in order to prove non negativity of $D_q(A:B|\{\Pi_k\})$. In fact, q-discord takes negative values for some states and for some q (see [28]) but it remains non negative for $q = 2$ [28], i.e.,

$$D_2(A:B|\Pi) \ge 0 \quad \text{as well as} \quad D_2(A:B) \ge 0$$

3. Final remarks

There is, however, another way to define total correlations given by (6). Starting from the Tsallis entropy

$$T_q(A|B_j) = \frac{1}{q-1}\Big[1 - \sum_i p_{i|j}^q\Big]$$

for classical conditional probability distribution

$$p_{i|j} := \frac{p_{ij}}{p_j^B},$$

where

$$p_j^B = \sum_i p_{ij}$$

one defines conditional Tsallis entropy as so-called q-expectation value with respect to the marginal probability distribution p_j^B, as [24]

$$\widetilde{T}_q = \sum_j u_j T_q(A|B_j),$$

where

$$u_j = \frac{(p_j^B)^q}{\sum_k (p_k^B)^q}.$$

See [29] for details. As a result one obtains

$$\widetilde{T}_q(A|B) = \frac{T_q(AB) - T_q(B)}{1 + (1-q)T_q(B)}$$

and

$$\begin{aligned}\widetilde{I}(A:B) &= T_q(A) - T_q(AB) \\ &= \frac{T_q(A) + T_q(B) - T_q(AB) + (1-q)T_q(A)T_q(B)}{1 + (1-q)T_q(B)}. \qquad (7)\end{aligned}$$

All the quantities on the right-hand side of (7) are well defined also in quantum case. Another q-discord based on (7) can be defined as

$$\widetilde{D}_q(A:B) := \widetilde{I}_q(A:B) - \widetilde{C}_q(A:B), \quad q > 1,$$

where in $\widetilde{C}_q$-calculation probabilities u_j were taken into account. Although there is no proof yet that $\widetilde{D}_q(A:B)$ is positive, it is the case at least for Werner states. In [29] it is proved that for

$$\rho_W = (1-c)\frac{I}{4} + c|\psi\rangle\langle\psi|, \quad 0 \le c \le 1$$

with $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, one obtains

$$\widetilde{D}_q(A:B) = \frac{1}{q-1}\Big[\frac{1}{2}\Big(\frac{1-c}{2}\Big)^q + \frac{1}{2}\Big(\frac{1+3c}{2}\Big)^q - \Big(\frac{1+c}{2}\Big)^q\Big] \ge 0.$$

The q-discord $\widetilde{D}_q(A:B)$ as a function of the parameter c is shown in Figure 1.

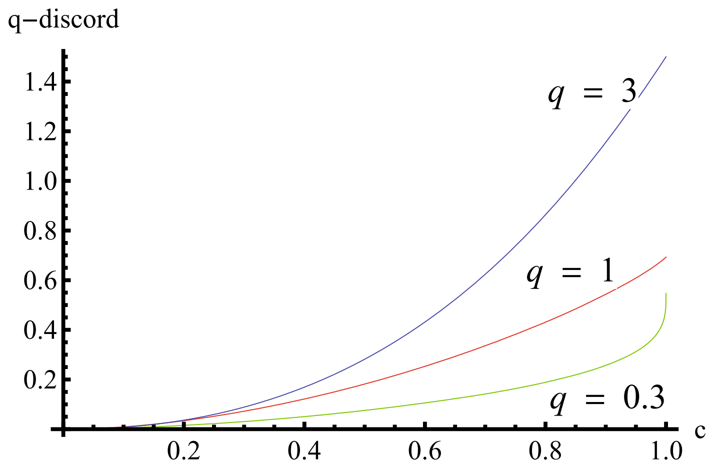

FIGURE 1. q-discord $\widetilde{D}_q(A : B)$ for Werner states

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Pseudopotentials via Moutard Transformations and Differential Geometry

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Abstract. Darboux-like (Moutard) and generalized Moutard transformations in two dimensions are applied to construct families of zero range potentials of scalar and matrix equations of stationary quantum mechanics. The statement about such functionals, defined by closed coordinate curves obtained by Ribokur-Moutard transforms is formulated. Their applications in physics and differential geometry of surfaces are discussed.

Mathematics Subject Classification (2010). 35A99; 14H70,81.

Keywords. Moutard transformations, Goursat equation, zero-range potential.

1. Introduction

Quite a number of problems in contemporary physics appear when continuous phenomena are joined with discrete one (discrete-continuous models). This concerns also point particles in quantum theory, mass tensors and Riemannian geometry in gravitation theory. The Dirac delta-function potential on the axis $x \in (-\infty, \infty)$ was first heuristically introduced by Fermi in a one-dimensional model. Its construction in the context of Neumann operator extension theory was understood in [1], see [2] for a review. The concept was realized as the theory of distributions on Schwartz space. A great number of applications of an advanced form of such potentials (zero-range potentials (ZRP) or pseudopotentials) appear in mesoscopic physics. Here it models objects whose dimension is small compared with the de Broglie wavelength of the electron. The generalization to the radial Schrödinger equation on the half-axis $r \in [0, \infty)$, started with a ZRP for s-states which was very successful in the application to scattering problems. From the point of a three-dimensional theory a mathematically rigorous formulation is given in [3]. Introducing the zero-range potential (ZRP) for two-dimensional problems needs special investigations [4].

We shall consider such problems from the point of view of a dressing technique for special cases of the Laplace equation, which allow a dressing procedure [5]. Such cases are known from the pioneering paper of Moutard [6]. More recent E. Ganzha applied it to an equation, equivalent to a Goursat equation [7]. Both equations have a direct link to the two-dimensional Schrödinger and Dirac equations.

As mentioned above, the Moutard and Goursat cases of the Laplace equations allow a kind of covariance statement which appeared already in [6, 8]. This was the starting point of the theory of Darboux transformations (DT). The DT in its original form [8] is a reduction of the Moutard transformation successfully applied by Darboux to the theory of surfaces.

One of the main observations is that the generalized ZRPs of the radial Schrödinger equation for arbitrary orbital quantum number l (GSRP), see, e.g., [3], appear as a result of iterated Darboux transformation in the context of radial Schrödinger equation theory. Such potentials are equivalent to boundary conditions, different for each l [9]. Namely, their three-dimensional description as pseudopotentials is studied in [3].

Two-dimensional ZRPs also may be obtained by DT-type transformations: the Moutard one and the generalized Moutard one for the Goursat case. The important feature of the MT is general for DT: the transform is parameterized by a pair of solutions of the equation and the transform vanishes if the solutions coincide. The Moutard equation (ME) is covariant with respect to the MT. It was studied in connection with central problems of classical differential geometry. More precisely, a chain of derivatives of solutions of the ME solves the system of Lamé equations for the Ribakur transformations [10].

In soliton theory the ME and GE enters the Lax pairs for nonlinear equations such as, for example, the Kadomtsev-Petviashvili and the Veselov-Novikov equation. This fact has important geometrical consequences as "integrable deformations of surfaces" [11].

In Section 2 we explain the general idea on an example of the radial Schrödinger equation along [9]. In Section 3, the Moutard transformation is used to define a chain of ZRP. The last section is devoted to the matrix ZRP problems of one of the two-dimensional two-component Dirac equation. The introduction of a pseudopotential by the generalized MT is traced.

2. General idea of ZRP introduction by dressing procedure

Let us consider a three-dimensional case of a so-called generalized ZRP [9]. Separation of variables yields the radial Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{l(l+1)}{2r^2}+u_l-E\right)\psi_l(r)=0. \tag{1}$$

where u_l are potentials for the partial waves. The equation (1) describes scattering of a particle with energy $E>0$ and momentum $k=\sqrt{2E}$. In the absence of a potential, partial shifts $\delta_l=0$ and partial waves can be expressed via Bessel functions with half-integer indices. Let us demonstrate how a generalized ZRP (GZRP) can be introduced by the DT. Thus, the spectral problem for GZRP is solved for any value k. On the other hand, the equation (1) is covariant with respect to the DT that yields the corresponding transformations of the potentials

A GZRP is equivalent to a boundary condition at the singularity point $r = 0$

$$\frac{1}{r^{l+1}\psi}\frac{\partial^{2l+1}}{\partial r^{2l+1}}\left(r^{l+1}\psi\right)\bigg|_{r=0} = -\frac{2^l l!}{(2l-1)!!}a_l^{2l+1}, \tag{2}$$

where we introduced $a_l^{2l+1} = -\frac{k^{2l+1}}{\tan \eta_l}$, with $s_l = \exp(2i\eta_l)$ being a scattering matrix. Such formulas are obtained by an application of an iterated DT to the zero potential solutions as follows. We start by choosing a spherical Bessel function as the seed solution $\psi_l(r) = Cj_l(kr)$ and apply Nth order Darboux transformation by taking spherical Hankel functions with specific parameters κ_m as prop functions $\varphi_m(r) = Ch_l^{(1)}(-i\kappa_m r)$, $m = 1, \ldots, N$. Crum's formula (e.g., [5]) gives the transformed solution

$$\psi_l^{[N]}(r) = C\frac{W(r\phi_1, \ldots, r\phi_N, r\psi_l)}{rW(r\phi_1, \ldots, r\phi_N)}. \tag{3}$$

The Wronskians can be computed if we consider the asymptotic behavior of the spherical functions at $r \to \infty$, the Wronskians turn into Vandermond determinants V, hence,

$$\psi_l^{[N]} = C\left[(-1)^l\frac{e^{ikr}}{kr}\frac{V(\kappa_1, \ldots, \kappa_N, ik)}{V(\kappa_1, \ldots, \kappa_N)} - \frac{e^{-ikr}}{kr}\frac{V(\kappa_1, \ldots, \kappa_N, -ik)}{V(\kappa_1, \ldots, \kappa_N)}\right]. \tag{4}$$

The Vandermond determinant can be computed by noticing that $k = -i\kappa_m$ (for $m = 1, \ldots, N$) are the roots of the polynomial with respect to k equation. This is obvious from the form of the matrix (replacing $ik \to \kappa_m$ yields that the determinant is zero due to the linear dependencies of the rows). Denoting $s_l = \prod_{m=1}^{N}(\kappa_m - ik)/(\kappa_m + ik)$, we recognize the asymptotics of spherical Hankel functions, hence

$$\psi_l^{[N]}(r) = C\left[s_l h_l^{(1)}(kr) - h_l^{(2)}(kr)\right]. \tag{5}$$

The effective potential corresponding to this solution tends to zero. Due to the asymptotic behaviour, we observe that the Darboux transformation does not change the behavior of the potential at $r \to \infty$, whereas the singular behavior at the origin is changed.

To sum up, the Darboux transformations significantly broaden the range of solvable potentials. In particular, they give a possibility to tune a free-space solution to potential scattering characteristics. Whilst the same transformation of the solution at the origin yields generalized zero-range potentials behavior.

3. Two-dimensional ZRP and Moutard transformation

Let us consider the Moutard equation

$$\psi_{\sigma\tau} + u(\sigma,\tau)\psi = 0 \ . \tag{6}$$

The Moutard transformation [6, 5] is a map of Darboux transformation type: it connects solutions and the coefficient $u(\sigma,\tau)$ of the equation (6) so that if φ and ψ are different solutions of it (6), then the solution of the twin equation with $\psi \to \psi[1]$ and $u(\sigma,\tau) \to u[1]$ can be constructed by the system

$$(\psi[1]\varphi)_\sigma = -\varphi^2(\psi\varphi^{-1})_\sigma,$$
$$(\psi[1]\varphi)_\tau = \varphi^2(\psi\varphi^{-1})_\tau.$$

In other words,

$$\psi[1] = \psi - \varphi\Omega(\varphi,\psi)/\Omega(\varphi,\varphi) \ , \tag{7}$$

where Ω is the integral of the exact differential form

$$d\Omega = \varphi\psi_\sigma d\sigma + \psi\varphi_\tau d\tau \ . \tag{8}$$

The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log\varphi)_{\sigma\tau} = -u + \varphi_\sigma\varphi_\tau/\varphi^2 \ . \tag{9}$$

Changing variables by the complex substitution $\sigma = x+iy, \tau = x-iy$ transforms (6) to a two-dimensional Schrödinger equation for x, y for potentials linked by $U(x,y) = -u(\sigma,\tau) + E$

$$-\frac{1}{4}[\psi_{xx} + \psi_{yy}] + U(x,y)\psi = E\psi \ . \tag{10}$$

The transformed potential is obtained via (9).

The explicit form of the ZRP depends on a choice of symmetry. For a cylindric symmetry [3], passing to polar coordinates $x = \rho\cos\phi, y = \rho\sin\phi$ and separating variables $\exp[i\nu\phi]R$ yields either R as solutions of the modified Bessel equation for $E = k^2 > 0$, or the Bessel equation for $E = -\kappa^2 < 0$. The case may be treated almost identically as in Section 2 by means of an iterated (multi-kink) MT, see the Wronskian formulas in [5].

We, however, develop the theory by the MT, extending it to more general symmetry, rewriting the (9) in polar coordinates

$$U[1] = U + \frac{1}{2}\Delta(\log\varphi) = U + \frac{1}{2}[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}](\log\varphi), \tag{11}$$

while $\psi[1]$ is the ψ transform by (7) with

$$\int d\Omega = \frac{1}{2}\int_{0,0}^{\rho,\phi}\left[(\psi\varphi)_\rho - i\ \frac{\varphi^2}{\rho}\left(\frac{\psi}{\varphi}\right)_\phi\right]d\rho + \left[(\psi\varphi)_\phi + i\rho\varphi^2\left(\frac{\psi}{\varphi}\right)_\rho\right]d\phi. \tag{12}$$

For $E = 0$, the Euler equation case in the ρ variable is obtained, and a general solution is $\psi = \sum_{\nu=-\infty}^{+\infty} c_n \exp[i\nu\phi]\rho^\nu$. To demonstrate it by an example, let us substitute the particular solutions $\varphi = \exp[i\nu\phi]\rho^\nu$ into the MT formulas (9). Direct differentiation prove a potential invariance $U[1] = U$. The same result gives the special case of $\nu = 0$, $\varphi = C \ln \rho + A$. Studying the case $E = 0$, take as ψ typical for scattering problems the free particle state $\psi = \exp[k_x \rho \cos\phi + k_y \rho \sin\phi]$. A choice of an integration curve in (12) yields

$$\int d\Omega = \frac{1}{2}(k_x - ik_y)\left(\int_o^\rho \rho^\nu e^{k_x z}\, dz + i\rho^{\nu+1}\int_0^\phi e^{[i(\nu+1)\beta + \rho k_x \cos\beta + \rho k_y \sin\beta]} d\beta\right).$$

Going to a vicinity of $\rho = 0$, approximating the integral and plugging it into the MT (7) gives for $\nu \neq -1$ a continuous function

$$\psi[1] = \exp[k_x \rho \cos\phi + k_y \rho \sin\phi] - \rho e^{-i\nu\phi}\frac{k_x - ik_y}{\nu + 1}\left(e^{i\phi(\nu+1)} + 1\right). \tag{13}$$

Consider the Hilbert space $H = L_2$ and a manifold of continuous functions $\psi \in M \subset H$. Applying Gauss theorem yields for a disk S inside a circumference L of small radius ϵ,

$$\lim_{L\to 0}\int_S \Delta\psi dS + 2\int_S \alpha\delta_2(\rho,\phi)\psi\rho d\rho d\phi = \lim_{L\to 0}\int_L (\vec{n}\cdot\nabla\psi)dL + 2\alpha\int_0^{2\pi}\psi(0,\phi)d\phi, \tag{14}$$

by definition of $\delta_2(\rho, \phi)$.

Generalizing to functions with possible singularity in $\rho = 0$, we arrive at a boundary condition for the solution (6) with zero potential of the form

$$\lim_{L\to 0}\frac{\int_L (\vec{n}\cdot grad\psi)\rho d\phi}{\int_0^{2\pi}\psi(\epsilon,\phi))d\phi} = 2\alpha. \tag{15}$$

Now we can formulate the approach to ZRP in two dimensions by the following algorithm. It is known that the set of iterated MT has an explicit link to Ribokur transformations. This defines solutions of the Lamé equations for coordinate systems [10], see also [12].

Generalizing (15), let us build a closed curve L as a coordinate line $\exists\epsilon > 0, a = a_0 \in [0,\epsilon], b \in [0,1]$ by means of such a construction and define the action of $\delta_2(a, b)$ by

Lemma. *The relation* $\int_S \delta_2(a,b)\psi(a,b)dS = \int_0^1 \psi(0,b)db$ *determines a distribution* $\delta_2(a, b) \in D$, *if* L *bounds a domain* S *(interior of* L*).*

For the proof it is enough to recall the isoperimetric inequality and the Jordan theorem; the functional linearity and continuity is obvious. Going to the set of coordinate systems a_n, b_n, numbered by the MT iteration number yields the

Theorem 1 (Main). *The set of distributions defined by*

$$\lim_{\epsilon\to 0}\frac{\int_0^1 (\vec{n}\cdot grad\psi)db_n}{\int_0^1 \psi(a_n,b_n)db_n} = 2\alpha \tag{16}$$

is dense in a vicinity of 0.

The proof is based on the lemma and the theorem of Ganzha on local completeness of iterated Moutard transformations [10].

4. Goursat equation, matrix ZRP and geometry of surfaces

Let us consider the Laplace equation

$$\psi_{\sigma\tau} + a(\sigma,\tau)\,\psi_\sigma + b(\sigma,\tau)\,\psi = 0. \tag{17}$$

The system

$$\psi_\sigma = p\chi, \quad \chi_\tau = p\psi, \tag{18}$$

is related directly to the Goursat equation

$$\psi_{\sigma\tau} = \frac{p_\tau}{p}\psi_\sigma + p^2\psi, \tag{19}$$

with the obvious constraint between a, b in (17); see [7], where a covariance with respect to a generalized MT was established. In [13], the matrix form of the problem for $\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \chi_1 & \chi_2 \end{pmatrix}$ was introduced in the variables ξ and η as:

$$\partial_\sigma = \partial_\eta - \partial_\xi, \qquad \partial_\tau = \partial_\eta + \partial_\xi,$$

and rewritten (18) in the form of 2x2 Dirac system:

$$\Psi_\eta = \sigma_3\Psi_\xi + U\Psi, \tag{20}$$

where $U = p(\xi,\eta)\sigma_1$. The functions $\psi_k = \psi_k(\xi,\eta)$, $\chi_k = \chi_k(\xi,\eta)$ with k=1,2 are particular solutions of (20) with some $p(\xi,\eta)$, and $\sigma_{1,3}$ are the Pauli matrices. Let $\Psi_1 \neq \Psi$ be a solution of the equation (20). We define a matrix function $\Xi \equiv \Psi_{1,\xi}\Psi_1^{-1}$. The equation (20) is covariant with respect to DT:

$$\Phi[1] = \Phi_\xi - \Xi\Phi, \qquad U[1] = U + [\sigma_3, \Xi]. \tag{21}$$

Let us consider a closed 1-form

$$d\Omega = \Phi\Psi d\xi + \Phi\sigma_3\Psi d\eta.$$

Lemma. *The form is exact if* Ψ *satisfies* (20) *and a* 2×2 *matrix function* Φ *solves the conjugate equation:*

$$\Phi_\eta = \Phi_\xi\sigma_3 - \Phi U. \tag{22}$$

The proof is by direct cross differentiation.

Theorem 2 ([13]). *One can verify by a substitution that* (22) *is covariant with respect to the transform if*

$$\Phi[+1] = \Omega(\Phi,\Psi_1)\Psi_1^{-1}. \tag{23}$$

Now we can alternatively affect U*, by the following transformation:*

$$U[+1,-1] = U + [\sigma_3, \Psi_1\Omega^{-1}\Phi]. \tag{24}$$

Relations (23), (24) we call a binary generalized Moutard transformation (BGMT).

Such a formalism gives a new possibility to define **ZRP for Dirac equation** via Darboux (21) or BGMT (23) transformation. The construction starts from a solution with a matrix potential U which directly relates to the equation (19) with constant p. Therefore we can use the solutions ψ_k of the Schrödinger equation (10) with $E = p^2$, constructed in the previous section. The matrices Ψ, Φ, are built from solutions ψ_k and $\chi_k = p^{-1}\psi_k$.

As **geometry** is concerned, the original Weierstrass formulas start with two arbitrary holomorphic functions of complex variables $z, \bar{z} \in C$ [12]. They yield an approach for constructing minimal surfaces. Generalization to the arbitrary mean curvature case was given by Kenmotsu [14] and Konopelchenko [11] in complex coordinates as in (6), τ, $\sigma = -\overline{\tau}$. Here p is a real-valued function and ψ or χ as solutions of (18) are complex-valued functions. We define three real-valued functions X_i, $i = 1, 2, 3$ which are the coordinates of a surface in $\mathbb{R}^3$: $X_1 + \imath X_2 = 2\imath \int_\Gamma \left(\overline{\psi^2} d\sigma' - \overline{\chi^2} d\tau'\right), X_3 = -2\int_\Gamma \left(\overline{\psi}\chi d\sigma' + \overline{\chi}\psi d\tau'\right)$, where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K and the mean curvature H yield:

$$ds^2 = 4N^2 d\tau d\sigma \;, \quad K = \frac{1}{N^2}\partial_\tau \partial_\sigma \ln N \;, \quad H = \frac{\sqrt{p}}{N} \;. \tag{25}$$

Here $N =\mid \psi \mid^2 + \mid \chi \mid^2$. Any analytic surface in $\mathbb{R}^3$ can be globally represented by X_i. As it is seen from the solutions nonzero N may yields zero p and hence zero mean curvature on a punctured surface [15].

Remark. Equation (20) is a spectral problem for the Davey-Stewartson (DS) and Boiti-Martina-Leon-Pempinelli (BMLP) equations and produce explicitly invertible Bäcklund auto-transformations. It also induces deformations of the correspondent surfaces following [11, 13].

5. Discussion and conclusion

The importance in applications of the pseudopotentials, introduced as distributions, lies in the possibility to solve multicenter scattering or eigenvalue problems [2]. The dressing procedure also may be applied to such multicenter pseudopotential. This gives additionally ability to approximate real interaction [5]. Technically it is applied to a combination of Green functions of the Schrödinger equation $\psi = \sum C_i G(|\vec{r} - \vec{r}_i|)$ and, next, substituting the result, to boundary conditions in each center $(\vec{r} = \vec{r}_i)$. The result is a set of algebraic equations. One of the interesting problems is related to quantum dots, randomly distributed by place and size, and modeled by a generalized ZRP. The theorem about a dense cover of the distribution space in a vicinity of a given point opens a way to developing new representations in potential theory. The problem of the matrix ZRP introduction is solved in an example of a two-dimensional Dirac equation. The idea of a dressing scheme is naturally generalized to other matrix problems as multi-channel scattering [5] or 4×4 matrix Dirac eigenvalue problem [16].

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Proving the Jacobi Identity the Hard Way

Kirill Mackenzie

Abstract. Vector fields on smooth manifolds may be regarded as derivations of the algebra of smooth functions, as infinitesimal generators of flows, or as sections of the tangent bundle. The last point of view leads to a formula for the bracket which is not used very often and in terms of which such a basic matter as proving the Jacobi identity seems difficult. We present a conceptually simple proof of the Jacobi identity in terms of this formulation.

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Keywords. Vector fields, Jacobi identity, double vector bundles, triple vector bundles.

There are three global formulas by which the bracket of vector fields can be calculated. Usually one interprets vector fields as derivations on the algebra of smooth functions, and the bracket is then the commutator of derivations. There is also the flow formula in which the bracket of vector fields is regarded as the Lie derivative of one field by the other.

Thirdly, for vector fields X, Y on a manifold M, and $m \in M$, there is:

$$[X, Y](m) = T(Y)(X(m)) - J(T(X)(Y(m))). \tag{1}$$

Here $J\colon T^2M \to T^2M$ is the canonical involution. This formula involves some abuse: the RHS is a vertical vector in $T_{Y(m)}(TM)$ and therefore can be identified with an element of T_mM. For convenience we refer to (1) as the 'section formula' for the bracket.

Formula (1) is much less widely used than the other two; one place in which it appears is [1, p. 297]. A proof can be extracted from [2, §3.4].

By using derivations the proof of the Jacobi identity can be done in one line; a few moments experimentation with (1) may leave the reader with the impression that using the section formula is unnatural and unwieldy. The purpose of this paper is to show that there is a diagrammatic proof, very easy to visualize, starting from (1), using double and triple vector bundles. This will be important in work elsewhere – since (1) uses only the tangent functor T and the canonical involution J, it can be formulated in more abstract settings.

The result is given, in a different language, in a paper of Nishimura [3]. I am grateful to Anders Kock who, many years ago, told me about this paper and gave me an offprint. The proof we give here in §4 is essentially a proof in coordinates. An intrinsic proof requires considerable length to be convincing and we may present it elsewhere.

1. Preliminaries on double vector bundles

Consider a square of vector bundle structures as in Figure 1(a). It is a *double vector bundle* if the operations which define the structure in $D \to B$ are morphisms with respect to the structures on $D \to A$ and $B \to M$. A detailed working-out of this definition is given in [2, Chap. 9].

For any vector bundle $q\colon A \to M$, applying the tangent functor to its operations gives a vector bundle structure on TA with base TM. This results in the double vector bundle shown in Figure 1(b). The projection is $T(q)$, the zero section is $T(0)$, and the addition $TA \times_{TM} TA \to TA$ is the tangent of the addition $A \times_M A \to A$.

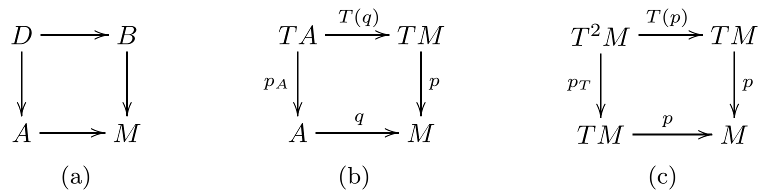

Figure 1. Note that these diagrams show individual structures; they should not be read as diagrams of morphisms. We denote the projections of tangent bundles by p with suffixes as needed.

In particular one may take A to be the tangent bundle TM and thus obtain the *double* or *iterated tangent bundle* $T^2M = T(TM)$ shown in Figure 1(c).

Return to the general case of Figure 1(a). The set of elements $d \in D$ which project to zero under both projections $D \to A$ and $D \to B$ is the *core of* D, denoted C; the two vector bundle structures on D coincide on C and make it a vector bundle on M. In the case $D = TA$ the core is A itself, identified with the vertical vectors along the zero section.

A *horizontal linear section* of a general double vector bundle D is a pair of sections $\xi\colon B \to D$ and $X\colon M \to A$ such that ξ is a morphism of vector bundles over X. One defines a *vertical linear section* (η, Y) of D in the analogous way.

Suppose given both a horizontal and a vertical linear section. Then, for $m \in M$, the projections of $\xi(Y(m))$ and $\eta(X(m))$ to both A and B coincide, and they therefore differ by a unique element of C, which we denote $w(\xi, \eta)(m)$. This defines a section $w(\xi, \eta)$ of C, which we call the *warp* of (ξ, X) and (η, Y).

For the proofs below, we need to formulate this precisely. Denote the additions in $D \to A$ and $D \to B$ by $\underset{A}{+}$ and $\underset{B}{+}$, and the subtractions likewise. Then

$$\eta(X(m)) \underset{A}{-} \xi(Y(m)) = w(\xi,\eta)(m) \underset{B}{+} 0^D_{X(m)}, \tag{2}$$

$$\eta(X(m)) \underset{B}{-} \xi(Y(m)) = w(\xi,\eta)(m) \underset{A}{+} 0^D_{Y(m)}, \tag{3}$$

where $0^D_{X(m)}$ is the zero element of $D \to A$ in the fibre over $X(m)$ and $0^D_{Y(m)}$ is the zero element of $D \to B$ in the fibre over $Y(m)$. We rely on the notation for elements to indicate the bundle.

Now (1) applied to T^2M shows that $[X, Y]$ is the warp of $(J \circ T(X), X)$ and $(T(Y), Y)$.

The map $J\colon T^2M \to T^2M$ is the *canonical involution*, which interchanges the two structures on T^2M; it is an isomorphism from the standard structure to the tangent prolongation structure and is an isomorphism of double vector bundles from T^2M to T^2M with the two structures interchanged.

Given a vector field X on M, applying the tangent functor gives $T(X)$, a section of $T(p)\colon T^2M \to TM$, as in Figure 1(b). Applying J gives a section of p_T; that is, a vector field on TM, called the *complete lift of* X. We write $\widetilde{X} = J \circ T(X)$.

2. Triple vector bundles

To deal with the terms in the Jacobi identity, we consider the *triple tangent bundle* as shown in Figure 2(a). The bottom face is the double tangent bundle of M, and the top face is the result of applying the tangent functor to it. The vertical arrows represent standard tangent bundle structures; we usually omit $p = p_M$.

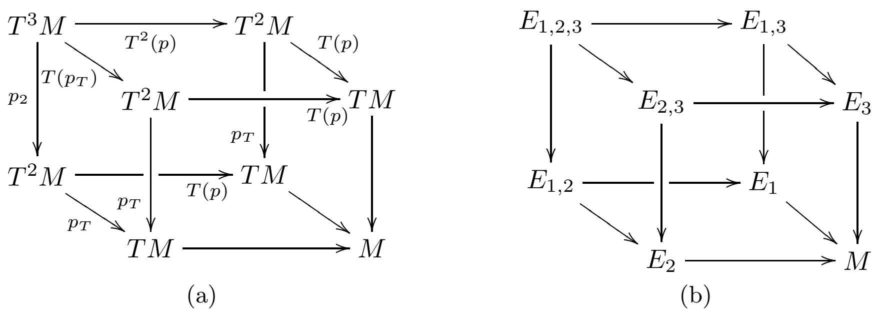

FIGURE 2. Oblique arrows should be read as coming out of the page.

We also need the general concept of triple vector bundle [4]. A *triple vector bundle* E is a cube of vector bundle structures, as shown in Figure 2(b), such that each face is a double vector bundle.

Each face of E has a core. For the three faces $E_{i,j}$ which involve M, we denote the cores by deleting the comma; thus the core of the bottom face is E_{12}. The core of the left face is denoted $E_{13,2}$, of the rear face is $E_{1,23}$, and of the top face is $E_{12,3}$. (This notation was developed in [4] to handle the n-fold case efficiently.)

The projection $E_{1,2,3} \to E_{1,2}$, together with the parallel projections, is a morphism of double vector bundles; that is, $E_{1,2,3} \to E_{1,2}$ is a morphism of vector bundles over both $E_{2,3} \to E_2$ and $E_{1,3} \to E_1$, and each of these is a morphism of vector bundles over $E_3 \to M$. It follows that $E_{1,2,3} \to E_{1,2}$ restricts to a map of the cores, $E_{12,3} \to E_{12}$. Further, the vector bundle structure of $E_{1,2,3}$ over $E_{1,2}$ restricts to give $E_{12,3}$ a vector bundle structure over E_{12}. And further, together with core vector bundle structures on $E_{12,3}$ and E_{12}, this forms a *core double vector bundle*, as shown in Figure 3(a). A detailed proof is given in [5].

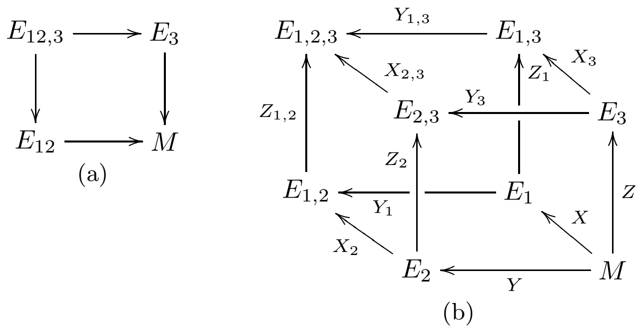

FIGURE 3.

The same construction can be carried out with the left-right structures and with the rear-front structures. Each of these core double vector bundles has a core, and the three cores coincide. This is the *ultracore of* E, denoted E_{123}.

Consider Figure 3(b). A *bottom-top linear double section of* E is a collection of sections

$$Z_{1,2}\colon\ E_{1,2} \to E_{1,2,3}, \quad Z_1\colon\ E_1 \to E_{1,3}, \quad Z_2\colon\ E_2 \to E_{2,3}, \quad Z\colon\ M \to E_3,$$

which form a morphism of double vector bundles. We write $Z_0\colon\ E_{12} \to E_{12,3}$ for the morphism of the cores. We define *left-right* and *front-rear linear double sections* in the corresponding ways.

Definition 1. A *grid on* E is a set of three double linear sections, one in each direction, as shown in Figure 3(b).

Consider a grid (X, Y, Z). In the top face, the warp of the sections $(X_{2,3}, X_3)$ and $(Y_{1,3}, Y_3)$ is a section of $E_{12,3} \to E_3$. In the bottom face, the warp of the sections (X_2, X) and (Y_1, Y) is a section of $E_{12} \to M$, and the two warps form a horizontal linear section of the core double vector bundle in Figure 3(a). Together with the linear section (Z_0, Z) this defines a pair of linear sections in the core double vector bundle, and the warp is the core section corresponding to

$$(Y_{1,3} \circ X_3 - X_{2,3} \circ Y_3) \circ Z - Z_0 \circ (Y_1 \circ X - X_2 \circ Y). \tag{4}$$

For brevity we denote the corresponding core section by $w(X, Y, Z)_{\text{top}}$. It is a section of the ultracore.

In the same way we obtain $w(X, Y, Z)_{\text{left}}$ and $w(X, Y, Z)_{\text{rear}}$, which are the sections of the ultracore corresponding respectively to

$$(X_{2,3} \circ Z_2 - Z_{1,2} \circ X_2) \circ Y - Y_0 \circ (X_3 \circ Z - Z_1 \circ X), \tag{5}$$

$$(Z_{1,2} \circ Y_1 - Y_{1,3} \circ Z_1) \circ X - X_0 \circ (Z_2 \circ Y - Y_3 \circ Z). \tag{6}$$

Theorem 1. *The sum of these three warps is zero,*

$$w(X, Y, Z)_{\text{top}} + w(X, Y, Z)_{\text{left}} + w(X, Y, Z)_{\text{rear}} = 0. \tag{7}$$

It is tempting to think that we need only replace X_0, Y_0, Z_0 by $X_{2,3}$, $Y_{3,1}$, $Z_{1,2}$, so that the second terms of (4), (5), (6) can be expanded out, and that then the sum will cancel. However, the subtraction signs in (4), (5), (6) refer to different structures.

We prove Theorem 1 in §4.

3. Proof of the Jacobi identity

First we apply Theorem 1 to the triple tangent bundle. Because we will need it below, we give a proof from (1) of the skew-symmetry of the bracket.

Lemma 2. *For vector fields X, Y on M, $[Y, X] = -[X, Y]$.*

Proof. Using (2) to state (1) carefully, we have

$$T(Y)(X(m)) \underset{T(p)}{-} J(T(X)(Y(m)) = [X, Y](m) + T(0)(Y(m)).$$

Here $\underset{T(p)}{-}$ denotes subtraction in the bundle with projection $T(p)$. Applying J to both sides, we have

$$J(T(Y)(X(m))) - T(X)(Y(m)) = [X, Y](m) \underset{T(p)}{+} 0_{Y(m)},$$

since J is the identity on the core of T^2M. Now the left-hand side is

$$-([Y, X](m) \underset{T(p)}{+} 0_{Y(m)}) = (-[Y, X](m)) \underset{T(p)}{+} 0_{Y(m)}. \qquad \square$$

Now let X, Y, Z be vector fields on M. They induce a grid, as shown in Figure 4.

In detail, the vector field X lifts across the bottom face to the complete lift $\widetilde{X}$. Across the right face it 'lifts' (though not to another vector field) to $T(X)$. The complete lift lifts in the same way across the left face to $T(\widetilde{X})$.

For Y we obtain $T(Y)$ twice and $T^2(Y)$. In the case of Z we obtain $\widetilde{Z}$ twice and the complete lift of the complete lift, which we denote $\widetilde{\widetilde{Z}}$.

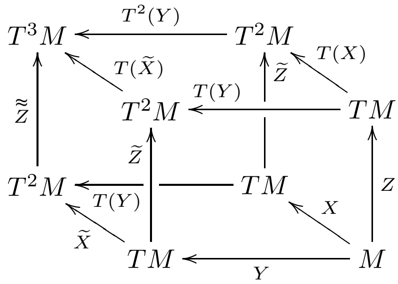

FIGURE 4.

In the bottom face the warp is of course $[X, Y]$. The top face is the result of applying the tangent functor to the bottom face, and the warp is accordingly $T([X, Y])$. The core of $\widetilde{\widetilde{Z}}$ is $\widetilde{Z}$ and so we have Figure 5(a). Thus

$$w(\widetilde{Z}, T([X, Y])) = [Z, [X, Y]]. \tag{8}$$

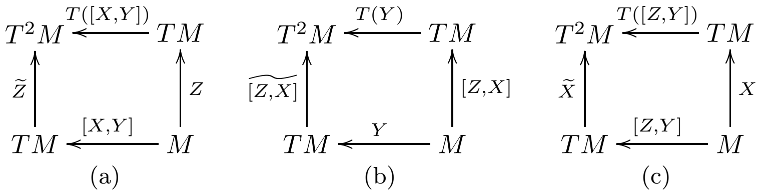

FIGURE 5.

From the right face we obtain $[Z, X]$. The left face is the double tangent bundle of the manifold TM and, as with any manifold, we have

$$T(\widetilde{X}) \circ \widetilde{Z} - J_T \circ T(\widetilde{Z}) \circ \widetilde{X} = [\widetilde{Z}, \widetilde{X}], \tag{9}$$

where J_T is the canonical involution for the manifold TM. The flows of a complete lift $\widetilde{X}$ are the tangents of the flows of X and it thereby follows that $[\widetilde{Z}, \widetilde{X}] = \widetilde{[Z, X]}$. The core section of $T^2(Y)$ is $T(Y)$ and so we have Figure 5(b). Substituting into (5) we have

$$\widetilde{[Z, X]} \circ Y - T(Y) \circ [Z, X] = -[[Z, X], Y] = [Y, [Z, X]]. \tag{10}$$

The rear face is not a double tangent bundle, but is rather Figure 1(b) for the vector bundle $A = T^2M \to TM$ with the tangent prolongation structure. We use canonical involutions to transform it into a double tangent bundle.

Figure 6(a) shows the rear face of Figure 4, and Figure 6(b) shows the tangent double vector bundle of the manifold TM. These are isomorphic under $T(J)\colon T^3M \to T^3M$, with J on the lower left manifolds and identities on the other two manifolds. That $T(p_T) \circ T(J) = T^2(p)$ follows trivially from $p_T \circ J = T(p)$, and $p_2 \circ T(J) = J \circ p_2$ is part of the statement that $T(J)$ is the tangent of J.

In Figure 6(b) we have, as with the previous case, that

$$w(\tilde{\tilde{\tilde{Z}}}, T(\tilde{Y})) = [\tilde{Z}, \tilde{Y}] = \widetilde{[Z, Y]}. \tag{11}$$

Now the core of the isomorphism between Figure 6(a) and (b) is J so the warp we actually want is

$$w(\tilde{\tilde{\tilde{Z}}}, T^2(Y)) = J \circ \widetilde{[Z, Y]} = T([Z, Y]). \tag{12}$$

We now have Figure 5(c) and substituting into (6) we have

$$-T([Y, Z]) \circ X + \tilde{X} \circ [Y, Z] = -[X, [Z, Y]] = [X, [Y, Z]]. \tag{13}$$

Together with Theorem 1 this proves that:

Corollary 3. *For vector fields X, Y, Z on M,*

$$[Z, [X, Y]] + [Y, [Z, X]] + [X, [Y, Z]] = 0.$$

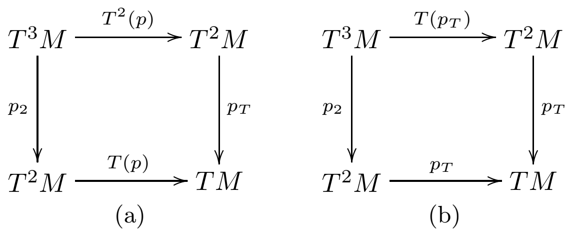

FIGURE 6.

4. Proof of Theorem 1

Given three vector bundles E_1, E_2, E_{12} on a manifold M, there is a double vector bundle structure on the pullback manifold $E := E_1 * E_2 * E_{12}$ (in this section $*$ denotes pullback over M). First, give E the inverse image vector bundle structure $q_1^!(E_2 \oplus E_{12})$ of the Whitney sum $E_2 \oplus E_{12}$ across the bundle projection $q_1\colon E_1 \to M$. Likewise, give E the inverse image vector bundle structure $q_2^!(E_1 \oplus E_{12})$. These two structures make E a double vector bundle with side bundles E_1 and E_2 and core bundle E_{12}; in [4] it is called the *decomposed double vector bundle with building bundles* E_1, E_2, E_{12}. Every double vector bundle is isomorphic to a decomposed double vector bundle [6], though not usually in any natural way.

In the same way, every triple vector bundle is isomorphic to a decomposed triple vector bundle, formed of seven building bundles E_1, E_2, E_3, E_{12}, E_{23}, E_{13}, E_{123}, with structures defined from inverse images of Whitney sums as in the double case. For details see [4].

It is sufficient to prove Theorem 1 for decomposed triple vector bundles. We first express linear sections in decomposed terms. Suppose given a linear section

$$X\colon M \to E_1,\ X_2\colon E_2 \to E_{1,2},\ X_3\colon E_3 \to E_{1,3},\ X_{2,3}\colon E_{2,3} \to E_{1,2,3}.$$

Then $X_2\colon E_2 \to E_{1,2}$ is a morphism of ordinary vector bundles over X and can be written as

$$X_2(e_2) = (X(m), e_2, \xi_2(e_2)) \in E_1 * E_2 * E_{12}$$

where $\xi_2\colon E_2 \to E_{12}$ is a vector bundle morphism over M. Likewise X_3 can be written as

$$X_3(e_3) = (X(m), e_3, \xi_3(e_3)) \in E_1 * E_3 * E_{13}$$

where $\xi_3\colon E_3 \to E_{13}$ is a vector bundle morphism over M.

Lastly, $X_{2,3}\colon E_{2,3} \to E_{1,2,3}$ is a morphism of decomposed double vector bundles over X_2, X_3 and X. In terms of the decomposition, $X_{2,3}$ is

$$X_{2,3}(e_2, e_3, e_{23}) = (X(m),\ e_2,\ e_3,\ \xi_2(e_2),\ e_{23},\ \xi_3(e_3),\ X_{23}(e_{23}) + \varphi_{23}(e_2, e_3)), \tag{14}$$

where ξ_2 and ξ_3 are as above, $X_{23}\colon E_{23} \to E_{123}$ is a morphism of ordinary vector bundles over M. and $\varphi_{23}\colon E_2 *_M E_3 \to E_{123}$ is bilinear. If $X_{2,3}$ were not a section of the bundle projection, there would be additional terms in (14).

Likewise, for linear sections $Y, Y_1, Y_3, Y_{1,3}$ and $Z, Z_1, Z_2, Z_{1,2}$ we have

$$Y_{1,3}(e_1, e_3, e_{13}) = (e_1,\ Y(m),\ e_3,\ \eta_1(e_1),\ \eta_3(e_3),\ e_{13},\ Y_{13}(e_{13}) + \psi_{13}(e_1, e_3)),$$

where $\eta_1\colon E_1 \to E_{12}$, $\eta_3\colon E_3 \to E_{23}$, $Y_{13}\colon E_{13} \to E_{123}$, $\psi_{13}\colon E_1 *_M E_3 \to E_{123}$, and

$$Z_{1,2}(e_1, e_2, e_{12}) = (e_1,\ e_2,\ Z(m),\ e_{12},\ \zeta_2(e_2),\ \zeta_1(e_1),\ Z_{12}(e_{12}) + \theta_{12}(e_1, e_2)),$$

where $\zeta_1\colon E_1 \to E_{13}$, $\zeta_2\colon E_2 \to E_{23}$, $Z_{12}\colon E_{12} \to E_{123}$ and $\theta_{12}\colon E_1 *_M E_2 \to E_{123}$.

We work out the core section corresponding to (4). First,

$$Y_1(X(m)) \underset{1}{-} X_2(Y(m)) = (X(m), 0^2, \eta_1(X(m)) - \xi_2(Y(m))).$$

Apply $Z_{1,2}$ to this. We get that $Z_{1,2}(X(m), e_2, (\eta_1 X - \xi_2 Y)(m))$ is

$$(X(m),\ 0^2,\ Z(m),\ (\eta_1 X - \xi_2 Y)(m),\ 0^{23},\ \zeta_1(X(m)),\ Z_{12}((\eta_1 X - \xi_2 Y)(m)))), \tag{15}$$

where 0^{23} is a zero of the ordinary vector bundle E_{23}.

Next we calculate

$$Y_{1,3}(X_3(Z(m))) \underset{1,3}{-} X_{2,3}(Y_3(Z(m))).$$

For the first term, $Y_{1,3}(X(m), Z(m), \zeta_3(Z(m)))$, we have

$$(X(m), Y(m), Z(m), \eta_1(X(m)), \eta_3(Z(m)), \xi_3(Z(m)),$$
$$Y_{13}(\xi_3(Z(m))) + \psi_{13}(X(m), Z(m))),$$

For the second term, $X_{2,3}(Y(m), Z(m), \eta_3(Z(m)))$, we have

$$(X(m),\, Y(m),\, Z(m),\, \xi_2(Y(m)),\, \eta_3(Z(m)),\, \xi_3(Z(m)),\\ X_{23}(\eta_3(Z(m))) + \varphi_{23}(Y(m), Z(m))),$$

We now subtract over $1, 3$; that is, we keep the E_1, E_3 and E_{13} entries fixed. We obtain

$$(X(m),\, 0^2,\, Z(m),\quad (\eta_1 X - \xi_2 Y)(m),\, 0^{23},\, \xi_3(Z(m)),\\ Y_{13}(\xi_3(Z(m))) + \psi_{13}(X(m), Z(m)) - X_{23}(\eta_3(Z(m))) - \varphi_{23}(Y(m), Z(m))). \quad (16)$$

Finally we subtract (16) from (15) over $1, 2$. This gives

$$(X(m),\, 0^2,\, 0^3,\quad (\eta_1 X - \xi_2 Y)(m),\, 0^{23},\, \zeta_1(X(m)) - \xi_3(Z(m)),\\ Z_{12}((\eta_1 X - \xi_2 Y)(m)) - (Y_{13}(\xi_3(Z(m))) + \psi_{13}(X(m), Z(m))\\ - X_{23}(\eta_3(Z(m))) - \varphi_{23}(Y(m), Z(m)))).$$

The final entry is the ultracore coordinate E_{123}, and simplifies to

$$X_{23}(\eta_3(Z(m))) - Y_{13}(\xi_3(Z(m))) + Z_{12}((\eta_1 X - \xi_2 Y)(m))\\ - \psi_{13}(X(m), Z(m)) + \varphi_{23}(Y(m), Z(m)). \quad (17)$$

In the same way we obtain the ultracore coordinates for (5) and (6). They are

$$Y_{13}(\zeta_1(X(m))) - Z_{12}(\eta_1(X(m))) + X_{23}((\zeta_2 Y - \eta_3 Z)(m))\\ - \theta_{12}(X(m), Y(m)) + \psi_{13}(X(m), Z(m)), \quad (18)$$

and

$$Z_{12}(\xi_2(Y(m))) - X_{23}(\zeta_2(Y(m))) + Y_{13}((\xi_3 Z - \zeta_1 X)(m))\\ - \varphi_{23}(Y(m), Z(m)) + \theta_{12}(X(m), Y(m)). \quad (19)$$

Adding (17), (18) and (19) gives zero. This concludes the proof.

5. Concluding remarks

It is not surprising that the LHS of the Jacobi identity can be interpreted in terms of the triple structure on T^3M; what does seem unexpected is that the identity itself follows from a general result which does not involve the specific properties of brackets or vector fields, but only the 'combinatorics' of the situation.

The calculation in this paper is special in several ways. In other situations, the warp (in one particular direction) of a grid may be of independent interest, and the fact that the three warps sum to zero may serve mainly as a check. The application of these ideas to other structures will be developed elsewhere.

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Löwner-Kufarev Evolution in the Segal-Wilson Grassmannian

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Abstract. We consider a homotopic evolution in the space of smooth shapes starting from the unit circle. Based on the Löwner-Kufarev equation we give a Hamiltonian formulation of this evolution and provide conservation laws. The symmetries of the evolution are given by the Virasoro algebra. The 'positive' Virasoro generators span the holomorphic part of the complexified tangent bundle over the space of conformal embeddings of the unit disk into the complex plane and smooth on the boundary. In the covariant formulation they are conserved along the Hamiltonian flow. The 'negative' Virasoro generators can be recovered by an iterative method making use of the canonical Poisson structure. We study an embedding of the Löwner-Kufarev trajectories into the Segal-Wilson Grassmannian. This gives a way to construct the τ-function, the Baker-Akhiezer function, and finally, to give a class of solutions to the KP equation.

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Keywords. Segal-Wilson Grassmannian, Virasoro algebra, univalent function, Löwner-Kufarev equation, Hamiltonian.

1. Introduction

This work is a short version of a plenary lecture given at the XXX Workshop on Geometric Methods in Physics held in Białowieża, June 26–July 02, 2011. The main idea of these short notes is to show that smooth shape evolution possesses an integrable structure. The first evidence of this was provided by the Laplacian growth (or the Hele-Shaw problem), where the process being dissipative possesses a countable number of conserved quantities, the harmonic (Richardson) moments,

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see [1]. Moreover, recently it became clear that the Laplacian growth is embedded in the dispersionless Toda hierarchy, see [2, 3]. An overview on Hele-Shaw flows and Laplacian growth one can find in [4]. The Laplacian growth represents a typical field problem, in which the evolution is defined by fixing an initial condition, the initial shape in this case. By *shape* we understand a smooth simple closed curve in the complex plane dividing it into two simply connected domains. The study of 2D shapes is one of the central problems in the field of applied sciences. A program of such study and its importance was summarized by Mumford at ICM 2002 in Beijing [5].

Another group of models, in which the evolution is governed by an infinite number of parameters, can be observed in infinite-dimensional controllable dynamical systems, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same algebraic structural background appears again in this type of models. We develop this viewpoint in the present paper.

One of the general approaches to the homotopic evolution of shapes starting from a canonical shape, the unit disk in our case, was provided by Löwner and Kufarev [6, 7, 8]. The shape evolution is described by a time-dependent conformal parametric map from the canonical domain onto the domain bounded by a shape at any fixed instant. In fact, these one-parameter conformal maps satisfy the Löwner-Kufarev differential equation, or an infinite-dimensional controllable system, for which the infinite number of conservation laws is given by the *Virasoro generators* in their covariant form.

Recently, Friedrich and Werner [9], and independently Bauer and Bernard [10], found relations between SLE (stochastic- or Schramm-Löwner evolution) and the highest weight representation of the Virasoro algebra. Moreover, Friedrich developed the Grassmannian approach to relate SLE with a singular highest weight representation of the Virasoro algebra in [11].

All above results encourage us to conclude that the *Virasoro algebra* is a common algebraic structural basis for these and possibly other types of contour dynamics and we present the development in this direction here. At the same time, the infinite number of conservation laws suggests a relation with exactly solvable models.

The geometry underlying classical integrable systems is reflected in Sato's [12] and Segal-Wilson's [13] constructions of the infinite-dimensional *Grassmannian* Gr. Based on the idea that the evolution of shapes in the plane is related to an evolution in a general universal space, the Segal-Wilson Grassmannian in our case, we provide an embedding of the Löwner-Kufarev evolution into a fiber bundle with the cotangent bundle over $\mathcal{F}_0$ as a base space, and with the smooth Grassmannian Gr_∞ as a typical fiber. Here $\mathcal{F}_0$ denotes the space of all conformal embeddings f of the unit disk into $\mathbb{C}$ normalized by $f(z) = z\,(1 + \sum_{n=1}^{\infty} c_n z^n)$ smooth on the boundary S^1, and under the *smooth Grassmannian* Gr_∞ we understand a dense subspace Gr_∞ of Gr.

We develop a Hamiltonian formalism for the Löwner-Kufarev evolution and define the Poisson structure. The main result gives an embedding of the Löwner-Kufarev evolution into the Segal-Wilson Grassmannian. We prove that the Virasoro generators in their covariant form are conserved along the Hamiltonian flow. Then we present the τ-function which gives the relation of the shape evolution to integrable systems. Then the powerful machinery of Segal-Wilson construction [13] can be switched on, and through the Baker-Akhiezer function and the definition of the KP flows one finds explicitly a new class of solutions to the KP hierarchy, see details in [14].

2. Löwner-Kufarev evolution

The pioneering idea of Löwner [7] in 1923 contained two main ingredients: subordination chains and semigroups of conformal maps. This far-reaching program was created in the hopes to solve the Bieberbach conjecture [15] and the final proof of this conjecture by de Branges [16] in 1984 was based on Löwner's parametric method. The modern form of this method is due to Kufarev [6] and Pommerenke [17, 8]. Omitting review over subordination chains we concentrate our attention on the other ingredient, i.e., on evolution families relating them to semigroups as in [18, 19, 8].

Let us consider a semigroup $\mathcal{P}$ of conformal univalent maps from the unit disk $\mathbb{D}$ into itself with superposition as a semigroup operation. This makes $\mathcal{P}$ a topological semigroup with respect of the topology of local uniform convergence on $\mathbb{D}$. We impose the natural normalization for such conformal maps $\Phi(z) = b_1 z + b_2 z^2 + \cdots$ about the origin, $b_1 > 0$. The unity of this semigroup is the identity map. A continuous homomorphism from $\mathbb{R}^+$ to $\mathcal{P}$ with a parameter $\tau \in \mathbb{R}^+$ gives a *semiflow* $\{\Phi^\tau\}_{\tau\in\mathbb{R}^+} \subset \mathcal{P}$ of conformal maps $\Phi^\tau : \mathbb{D} \to \Omega \subseteq \mathbb{D}$, satisfying the properties

- $\Phi^0 = id$;
- $\Phi^{\tau+s} = \Phi^s \circ \Phi^\tau$;
- $\Phi^\tau(z) \to z$ locally uniformly in $\mathbb{D}$ as $\tau \to 0$.

In particular, $\Phi^\tau(z) = b_1(\tau)z + b_2(\tau)z^2 + \cdots$, and $b_1(0) = 1$. This semi-flow is generated by a vector field $v(z)$ if for each $z \in \mathbb{D}$ the function $w = \Phi^\tau(z), \tau \geq 0$ is a solution to an autonomous differential equation $dw/d\tau = v(w)$, with the initial condition $w(z,\tau)\Big|_{\tau=0} = z$. This vector field, called infinitesimal generator, is given by $v(z) = -zp(z)$ where $p(z)$ is a regular Carathéodory function in the unit disk, with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$.

We call a subset $\Phi^{t,s}$ of $\mathcal{P}$, $0 \leq s \leq t$ an *evolution family* if

- $\Phi^{t,t} = id$;
- $\Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s}$, for $0 \leq s \leq r \leq t$;
- $\Phi^{t,s}(z) \to z$ locally uniformly in $\mathbb{D}$ as $t - s \to 0$.

In particular, if Φ^τ is a one-parameter semiflow, then Φ^{t-s} is an evolution family. Given an evolution family $\{\Phi^{t,s}\}_{t,s}$, every function $\Phi^{t,s}$ is univalent and there exists an essentially unique infinitesimal generator, called the Herglotz vector field $H(z,t)$, such that

$$\frac{d\Phi^{t,s}(z)}{dt} = H(\Phi^{t,s}(z), t), \tag{1}$$

where the function H is given by $H(z,t) = -zp(z,t)$ with a Carathéodory function p for almost all $t \geq 0$. The converse is also true. Solving equation (1) with the initial condition $\Phi^{s,s} = \text{id}$, we obtain an evolution family. In particular, we can consider the situation when $s = 0$. Let $f(z,t) = e^t w(z,t)$. A remarkable property of evolution families is that any conformal embedding f of the unit disk $\mathbb{D}$ to $\mathbb{C}$ normalized by $f(z) = z + c_1 z^2 + \cdots$ in $\mathbb{D}$ can be obtained as a one-parameter homotopy from the identity map, i.e.,

$$f(z) = \lim_{t\to\infty} f(z,t) = \lim_{t\to\infty} e^t w(z,t),$$

where the function

$$w(z,t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right),$$

solves the Cauchy problem for the Löwner-Kufarev ODE

$$\frac{dw}{dt} = -wp(w,t), \quad w(z,t)\bigg|_{t=0} = z, \tag{2}$$

and with the function $p(z,t) = 1 + p_1(t)z + \cdots$ which is holomorphic in $\mathbb{D}$ for almost all $t \in [0,\infty)$, measurable with respect to $t \in [0,\infty)$ for any fixed $z \in \mathbb{D}$, and such that $\operatorname{Re} p > 0$ in $\mathbb{D}$, see [8]. The function $w(z,t) = \Phi^{t,0}(z)$ is univalent and maps $\mathbb{D}$ into $\mathbb{D}$.

Lemma 1. *Let the function $w(z,t)$ be a solution to the Cauchy problem* (2)*. If the driving function $p(\cdot,t)$, being from the Carathéodory class for almost all $t \geq 0$, is C^∞ smooth in the closure $\hat{\mathbb{D}}$ of the unit disk $\mathbb{D}$ and summable with respect to t, then the boundaries of the domains $\Omega(t) = w(\mathbb{D},t) \subset \mathbb{D}$ are smooth for all t and $w(\cdot,t)$ extended to S^1 is injective on S^1.*

Lemma 2. *With the above notations let $f(z) \in \mathcal{F}_0$. Then there exists a function $p(\cdot,t)$ from the Carathéodory class for almost all $t \geq 0$, and C^∞ smooth in $\hat{\mathbb{D}}$, such that $f(z) = \lim_{t\to\infty} f(z,t)$ is the final point of the Löwner-Kufarev trajectory with the driving term $p(z,t)$.*

3. Hamiltonian formalism

Let the driving term $p(z,t)$ in the Löwner-Kufarev ODE (2) be from the Carathéodory class for almost all $t \geq 0$, C^∞-smooth in $\hat{\mathbb{D}}$, and summable with respect to t as in Lemma 1. Then the domains $\Omega(t) = f(\mathbb{D},t) = e^t w(\mathbb{D},t)$ have smooth boundaries

$\partial\Omega(t)$ and the function f is injective on S^1, i.e.; $f \in \mathcal{F}_0$. So the Löwner-Kufarev equation can be extended to the closed unit disk $\hat{\mathbb{D}} = \mathbb{D} \cup S^1$.

Let us consider the sections ψ of $T^*\mathcal{F}_0 \otimes \mathbb{C}$, that are from the class $C^\infty_{\|\cdot\|_2}$ of smooth complex-valued functions $S^1 \to \mathbb{C}$ endowed with L^2 norm,

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^{k-1}, \quad |z| = 1.$$

We also introduce the space of observables on $T^*\mathcal{F}_0 \otimes \mathbb{C}$, given by integral functionals

$$\mathcal{R}(f, \bar{\psi}, t) = \frac{1}{2\pi} \int_{z\in S^1} r(f(z), \bar{\psi}(z), t) \frac{dz}{iz},$$

where the function $r(\xi, \eta, t)$ is smooth in variables ξ, η and measurable in t.

We define a special observable, the time-dependent pseudo-Hamiltonian $\mathcal{H}$, by

$$\mathcal{H}(f, \bar{\psi}, p, t) = \frac{1}{2\pi} \int_{z\in S^1} \bar{z}^2 f(z,t)(1 - p(e^{-t} f(z,t), t)) \bar{\psi}(z,t) \frac{dz}{iz}, \tag{3}$$

with the driving function (control) $p(z,t)$ satisfying the above properties. The Poisson structure on the space of observables is given by the canonical brackets

$$\{\mathcal{R}_1, \mathcal{R}_2\} = 2\pi \int_{z\in S^1} z^2 \left(\frac{\delta \mathcal{R}_1}{\delta f} \frac{\delta \mathcal{R}_2}{\delta \bar{\psi}} - \frac{\delta \mathcal{R}_1}{\delta \bar{\psi}} \frac{\delta \mathcal{R}_2}{\delta f} \right) \frac{dz}{iz},$$

where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta \bar{\psi}}$ are the variational derivatives, $\frac{\delta}{\delta f}\mathcal{R} = \frac{1}{2\pi}\frac{\partial}{\partial f} r$, $\frac{\delta}{\delta \bar{\psi}}\mathcal{R} = \frac{1}{2\pi}\frac{\partial}{\partial \bar{\psi}} r$.

Representing the coefficients c_n and $\bar{\psi}_m$ of f and $\bar{\psi}$ as integral functionals

$$c_n = \frac{1}{2\pi} \int_{z\in S^1} \bar{z}^{n+1} f(z,t) \frac{dz}{iz}, \quad \bar{\psi}_m = \frac{1}{2\pi} \int_{z\in S^1} z^{m-1} \bar{\psi}(z,t) \frac{dz}{iz},$$

$n \in \mathbb{N}$, $m \in \mathbb{Z}$, we obtain $\{c_n, \bar{\psi}_m\} = \delta_{n,m}$, $\{c_n, c_k\} = 0$, and $\{\bar{\psi}_l, \bar{\psi}_m\} = 0$, where $n, k \in \mathbb{N}$, $l, m \in \mathbb{Z}$.

The infinite-dimensional Hamiltonian system is written as

$$\frac{dc_k}{dt} = \{c_k, \mathcal{H}\}, \tag{4}$$

$$\frac{d\bar{\psi}_k}{dt} = \{\bar{\psi}_k, \mathcal{H}\}, \tag{5}$$

where $k \in \mathbb{Z}$ and $c_0 = c_{-1} = c_{-2} = \cdots = 0$, or equivalently, multiplying by corresponding powers of z and summing up,

$$\frac{df(z,t)}{dt} = f(1 - p(e^{-t} f, t)) = 2\pi \frac{\delta \mathcal{H}}{\delta \bar{\psi}} z^2 = \{f, \mathcal{H}\}, \tag{6}$$

$$\frac{d\bar{\psi}}{dt} = -(1 - p(e^{-t} f, t) - e^{-t} f p'(e^{-t} f, t)) \bar{\psi} = -2\pi \frac{\delta \mathcal{H}}{\delta f} z^2 = \{\bar{\psi}, \mathcal{H}\}, \tag{7}$$

where $z \in S^1$. So the phase coordinates $(f, \bar{\psi})$ play the role of the canonical Hamiltonian pair. Observe that the equation (6) is the Löwner-Kufarev equation (2) for the function $f = e^t w$.

Let us set up the *generating function* $\mathcal{G}(z)=\sum_{k\in\mathbb{Z}}\mathcal{G}_k z^{k-1}$, such that

$$\bar{\mathcal{G}}(z):=f'(z,t)\bar{\psi}(z,t).$$

Consider the 'non-positive' $(\bar{\mathcal{G}}(z))_{\leq 0}$ and 'positive' $(\bar{\mathcal{G}}(z))_{>0}$ parts of the Laurent series for $\bar{\mathcal{G}}(z)$:

$$(\bar{\mathcal{G}}(z))_{\leq 0}=(\bar{\psi}_1+2c_1\bar{\psi}_2+3c_2\bar{\psi}_3+\cdots)+(\bar{\psi}_2+2c_1\bar{\psi}_3+\cdots)z^{-1}+\cdots=\sum_{k=0}^{\infty}\bar{\mathcal{G}}_{k+1}z^{-k}.$$

$$(\bar{\mathcal{G}}(z))_{>0}=(\bar{\psi}_0+2c_1\bar{\psi}_1+3c_2\bar{\psi}_2+\cdots)z+(\bar{\psi}_{-1}+2c_1\bar{\psi}_0+3c_2\bar{\psi}_1\cdots)z^2+\cdots=\sum_{k=1}^{\infty}\bar{\mathcal{G}}_{-k+1}z^{k}.$$

Proposition 1. *Let the driving term $p(z,t)$ in the Löwner-Kufarev ODE be from the Carathéodory class for almost all $t\geq 0$, C^∞-smooth in $\hat{\mathbb{D}}$, and summable with respect to t. The functions $\mathcal{G}(z)$, $(\mathcal{G}(z))_{<0}$, $(\mathcal{G}(z))_{\geq 0}$, and all coefficients $\mathcal{G}_n$ are time-independent for all $z\in S^1$.*

Proof. It is sufficient to check the equality $\dot{\bar{\mathcal{G}}}=\{\bar{\mathcal{G}},\mathcal{H}\}=0$ for the function $\mathcal{G}$, and then, the same holds for the coefficients of the Laurent series for $\mathcal{G}$. □

Proposition 2. *The conjugates $\bar{\mathcal{G}}_k$, $k=1,2,\ldots$, to the coefficients of the generating function satisfy the Witt commutation relation $\{\bar{\mathcal{G}}_m,\bar{\mathcal{G}}_n\}=(n-m)\bar{\mathcal{G}}_{n+m}$ for $n,m\geq 1$, with respect to our Poisson structure.*

The isomorphism $\iota:\ \bar{\psi}_k\to\partial_k=\frac{\partial}{\partial c_k}$, $k>0$, is a Lie algebra isomorphism $(T_f^{*(0,1)}\mathcal{F}_0,\{\ ,\ \})\to(T_f^{(1,0)}\mathcal{F}_0,[\ ,\ ])$. It makes a correspondence between the conjugates $\bar{\mathcal{G}}_n$ of the coefficients $\mathcal{G}_n$ of $(\mathcal{G}(z))_{\geq 0}$ at the point $(f,\bar{\psi})$ and the Kirillov vectors $L_n[f]=\partial_n+\sum\limits_{k=1}^{\infty}(k+1)c_k\partial_{n+k}$, $n\in\mathbb{N}$, see [20]. Both satisfy the Witt commutation relations

$$[L_n,L_m]=(m-n)L_{n+m}.$$

4. Curves in Grassmannian

Let us recall, that the underlying space for the universal smooth Grassmannian $\mathrm{Gr}_\infty(H)$ is $H=C^\infty_{\|\cdot\|_2}(S^1)$ with the canonical L^2 inner product of functions defined on the unit circle. Its natural polarization

$$H_+=\mathrm{span}_H\{1,z,z^2,z^3,\ldots\},\qquad H_-=\mathrm{span}_H\{z^{-1},z^{-2},\ldots\},$$

was introduced before. The pseudo-Hamiltonian $\mathcal{H}(f,\bar{\psi},t)$ is defined for an arbitrary $\psi\in L^2(S^1)$, but we consider only smooth solutions of the Hamiltonian system, therefore, $\psi\in H$. We identify this space with the dense subspace of $T_f^*\mathcal{F}_0\otimes\mathbb{C}$,

$f \in \mathcal{F}_0$. The generating function $\mathcal{G}$ defines a linear map $\bar{\mathcal{G}}$ from the dense subspace of $T^*_f\mathcal{F}_0 \otimes \mathbb{C}$ to H, which being written in a block matrix form becomes

$$\begin{pmatrix} \bar{\mathcal{G}}_{>0} \\ \bar{\mathcal{G}}_{\leq 0} \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{>0} \\ \bar{\psi}_{\leq 0} \end{pmatrix}, \tag{8}$$

where

$$\begin{pmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{1,1} \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & 7c_6 & \cdots \\ \cdots & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & 6c_5 & \cdots \\ \cdots & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & 5c_4 & \cdots \\ \hline \cdots & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & 4c_3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & 3c_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 2c_1 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right).$$

Proposition 3. *The operator $C_{1,1}\colon H_+ \to H_+$ is invertible.*

The generating function also defines a map $\mathcal{G}\colon T^*\mathcal{F}_0 \otimes \mathbb{C} \to H$ by

$$T^*\mathcal{F}_0 \otimes \mathbb{C} \ni (f(z), \psi(z)) \mapsto \mathcal{G} = \bar{f}'(z)\psi(z) \in H.$$

Observe that any solution $\big(f(z,t), \bar{\psi}(z,t)\big)$ of the Hamiltonian system is mapped into a single point of the space H, since all $\mathcal{G}_k$, $k \in \mathbb{Z}$ are time-independent by Proposition 1.

Consider a bundle $\pi\colon \mathcal{B} \to T^*\mathcal{F}_0 \otimes \mathbb{C}$ with a typical fiber isomorphic to $\mathrm{Gr}_\infty(H)$. We are aimed at construction of a curve $\Gamma\colon [0,T] \to \mathcal{B}$ that is traced by the solutions to the Hamiltonian system, or in other words, by the Löwner-Kufarev evolution. The curve Γ will have the form

$$\Gamma(t) = \Big(f(z,t), \psi(z,t), W_{T_n}(t)\Big)$$

in the local trivialization. Here W_{T_n} is the graph of a finite rank operator $T_n\colon H_+ \to H_-$, such that W_{T_n} belongs to the connected component of U_{H_+} of virtual dimension 0. In other words, we build a hierarchy of finite rank operators $T_n\colon H_+ \to H_-$, $n \in \mathbb{Z}^+$, whose graphs in the neighborhood U_{H_+} of the point $H_+ \in \mathrm{Gr}_\infty(H)$ are

$$T_n((\mathcal{G}(z))_{>0}) = T_n(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) = \begin{cases} G_0(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) \\ G_{-1}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots) \\ \cdots \\ G_{-n+1}(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k, \dots), \end{cases}$$

with $G_0 z^{-1} + G_{-1} z^{-2} + \cdots + G_{-n+1} z^{-n} \in H_-$. Let us denote by $G_k = \mathcal{G}_k$, $k \in \mathbb{N}$. The elements $G_0, G_{-1}, G_{-2}, \dots$ are constructed so that all $\{\bar{G}_k\}_{k=-n+1}^{\infty}$ satisfy the truncated Witt commutation relations

$$\{\bar{G}_k, \bar{G}_l\}_n = \begin{cases} (l-k)\bar{G}_{k+l}, & \text{for } k+l \geq -n+1, \\ 0, & \text{otherwise}, \end{cases}$$

and are related to Kirillov's vector fields [20] under the isomorphism ι. The projective limit as $n \leftarrow \infty$ recovers the whole Witt algebra and the Witt commutation relations. Then the operators T_n such that their conjugates are $\bar{T}_n = (\widetilde{B}^{(n)} + C_{2,1}^{(n)}) \circ C_{1,1}^{-1}$, are operators from H_+ to H_- of finite rank and their graphs $W_{T_n} = (\mathrm{id} + T_n)(H_+)$ are elements of the component of virtual dimension 0 in $\mathrm{Gr}_\infty(H)$. We can construct a basis $\{e_0, e_1, e_2, \dots\}$ in W_{T_n} as a set of Laurent polynomials defined by means of operators T_n and $\bar{C}_{1,1}$ as a mapping

$$\{\psi_1, \psi_2, \dots\} \xrightarrow{\bar{C}_{1,1}} \{G_1, G_2, \dots\} \xrightarrow{\mathrm{id} + T_n} \{G_{-n+1}, G_{-n+2}, \dots, G_0, G_1, G_2, \dots\},$$

of the canonical basis $\{1,0,0,\dots\}$, $\{0,1,0,\dots\}$, $\{0,0,1,\dots\}$,...

Let us formulate the result as the following main statement.

Proposition 4. *The operator T_n defines a graph $W_{T_n} = \mathrm{span}\{e_0, e_1, e_2, \dots\}$ in the Grassmannian Gr_∞ of virtual dimension 0. Given any*

$$\psi = \sum_{k=0}^{\infty} \psi_{k+1} z^k \in H_+ \subset H,$$

the function

$$G(z) = \sum_{k=-n}^{\infty} G_{k+1} z^k = \sum_{k=0}^{\infty} \psi_{k+1} e_k,$$

is an element of W_{T_n}.

Proposition 5. *In the autonomous case of the Cauchy problem* (2)*, when the function $p(z,t)$ does not depend on t, the pseudo-Hamiltonian $\mathcal{H}$ plays the role of time-dependent energy and $\mathcal{H}(t) = \bar{G}_0(t) + const$, where $\bar{G}_0\big|_{t=0} = 0$. The constant is defined as $\sum_{n=1}^{\infty} p_k \bar{\psi}_k(0)$.*

Remark 1. *The Virasoro generator L_0 plays the role of the energy functional in CFT. In the view of the isomorphism ι, the observable $\bar{G}_0 = \iota^{-1}(L_0)$ plays an analogous role.*

Thus, we constructed a countable family of curves $\Gamma_n \colon [0,T] \to \mathcal{B}$ in the trivial bundle $\mathcal{B} = T^*\mathcal{F}_0 \otimes \mathbb{C} \times \mathrm{Gr}_\infty(H)$, such that the curve Γ_n admits the form $\Gamma_n(t) = \Big(f(z,t), \psi(z,t), W_{T_n}(t)\Big)$, for $t \in [0,T]$ in the local trivialization. Here $\big(f(z,t), \bar{\psi}(z,t)\big)$ is the solution of the Hamiltonian system (4)–(5). Each operator $T_n(t) \colon H_+ \to H_-$ that maps $\mathcal{G}_{>0}$ to

$$\big(G_0(t), G_{-1}(t), \dots, G_{-n+1}(t)\big)$$

defined for any $t \in [0, T]$, $n = 1, 2, \ldots$, is of finite rank and its graph $W_{T_n}(t)$ is a point in $\mathrm{Gr}_\infty(H)$ for any t. The graphs W_{T_n} belong to the connected component of the virtual dimension 0 for every time $t \in [0, T]$ and for fixed n. The coordinates $(G_{-n+1}, \ldots, G_{-2}, G_{-1}, G_0, G_1, G_2, \ldots)$ of a point in the graph W_{T_n} considered as a function of ψ are isomorphic to the Kirillov vector fields

$$(L_{-n+1}, \ldots, L_{-2}, L_{-1}, L_0, L_1, L_1, L_2, \ldots)$$

under the isomorphism ι.

5. τ-function

Remind that any function g holomorphic in the unit disc, non vanishing on the boundary and normalized by $g(0) = 1$ defines the multiplication operator $g\varphi$, $\varphi(z) = \sum_{k\in\mathbb{Z}} \varphi_k z^k$, that can be written in the matrix form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \varphi_{\geq 0} \\ \varphi_{<0} \end{pmatrix}. \tag{9}$$

All these upper triangular matrices form a subgroup GL^+_{res} of the group of automorphisms GL_{res} of the Grassmannian $\mathrm{Gr}_\infty(H)$.

With any function g and any graph W_{T_n} constructed in the previous section (which is transverse to H_-) we can relate the τ-function $\tau_{W_{T_n}}(g)$ by the following formula

$$\tau_{W_{T_n}}(g) = \det(1 + a^{-1} b T_n),$$

where a, b are the blocks in the multiplication operator generated by g^{-1}. If we write the function g in the form $g(z) = \exp(\sum_{n=1}^\infty t_n z^n) = 1 + \sum_{k=1}^\infty S_k(\mathbf{t}) z^k$, where the coefficients $S_k(\mathbf{t})$ are the homogeneous elementary Schur polynomials, then the coefficients $\mathbf{t} = (t_1, t_2, \ldots)$ are called generalized times. For any fixed W_{T_n} we get an orbit in $\mathrm{Gr}_\infty(H)$ of curves Γ_n constructed in the previous section under the action of the elements of the subgroup GL^+_{res} defined by the function g. On the other hand, the τ-function defines a section in the determinant bundle over $\mathrm{Gr}_\infty(H)$ for any fixed $f \in \mathcal{F}_0$ at each point of the curve Γ_n.

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The pre-Maxwell Equations

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Abstract. For space-time we consider linear differential equations from which the Maxwell equations follow. These equations we call pre-Maxwell equations. If the pre-Maxwell equations hold then the tensor of the electromagnetic field F, the four-current J and the energy-momentum tensor T will have some interesting properties which we present. Finally we identify integrals of the pre-Maxwell equations in flat and in de Sitter space-time.

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1. The definition of the pre-Maxwell equations

Let (M, g) be a space-time of General Relativity, which means M is a smooth four-dimensional manifold endowed with a metric tensor g of Lorentzian signature $(-\,+\,+\,+)$ with respect to which M is time oriented and with Levi-Civita connection ∇ (see [1]).

Suppose that ω is the electromagnetic field 2-form. In terms of local coordinates $\{x^0, x^1, x^2, x^3\}$ the electromagnetic field 2-form ω may be written as $\omega = \frac{1}{2}\, F_{ij}\, dx^i \wedge dx^j$, where $F_{ij} = F(\partial_i, \partial_j)$ are local components of the electromagnetic field tensor F for $\partial_k = \partial/\partial x^k$ and $i, j, k = 0, 1, 2, 3$.

It is well known that ω is a closed 2-form, i.e.,

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0,$$

and obeys the Maxwell equations in the presence of a charge (see [1])

$$\nabla^k F_{kj} = 4\,\pi\, J_j$$

for local coordinates J_j of the current 4-vector J.

Definition 1. The following equations

$$\nabla_i F_{jk} = 4/3\ \pi\,(J_j g_{ik} - J_k g_{ij}) \tag{1}$$

for $g_{ij} = g(\partial_i, \partial_j)$ are called the *pre-Maxwell equations.*

From the pre-Maxwell equations (1) we automatically obtain the Maxwell equations in the presence of a charge.

Remark 1. If the electromagnetic field tensor F satisfies equations (1) then it is a *closed skew-symmetric conformal Killing* 2*-tensor* (see [2]).

2. Properties of space-times admitting the pre-Maxwell equations

If the pre-Maxwell equations (1) hold then the tensor F of electromagnetic field, current 4-vector J and the energy-momentum tensor T have the following properties.

Proposition 2. *The electromagnetic field tensor F is parallel along integrals curves of the current* 4*-vector J.*

Proof. From (1) we have $J^k \nabla_k F_{ij} = 0$ which means $\nabla_J F = 0$. Hence the tensor F is parallel along integral curves of the 4-vector J. □

Proposition 3. *The current* 4*-vector J is a Killing vector if and only if (M, g) is an Einstein manifold, i.e.,* $\mathrm{Ric} = \lambda\, g$ *for* $\lambda = \mathrm{const}$.

Proof. We use the following the Ricci identities for the electromagnetic field tensor F in coordinate form (see [3])

$$\nabla_i \nabla_j F_{kl} - \nabla_j \nabla_i F_{kl} = -F_{ml} R^m_{kij} - F_{km} R^m_{lij} \tag{2}$$

for local components R^m_{kij} of the curvature tensor R of (M, g). Substituting (1) in the Ricci identities we have

$$\nabla_i J_k = \frac{3}{8\pi}\ (F^{ml} R_{mlki} - F_{km} R^m_i). \tag{3}$$

for components $R_{ij} = R^m_{imj}$ of the Ricci tensor Ric (see [3]). □

A vector field X on (M, g) is called a Killing vector if the local flows generated by X act by a 1-parameter group of isometric transformations (see [3]). This translates into the following simple equations $\nabla_i X_k + \nabla_k X_i = 0$. The current 4-vector J generates a 1-parameter group of isometric transformations if and only if $F_{km} R^m_i + F_{im} R^m_k = 0$. From these identities we conclude that $\mathrm{Ric} = \lambda\, g$ for $\lambda = \mathrm{const}$.

Proposition 4. *Let (M, g) be a de Sitter space-time, i.e., a Lorentzian manifold of constant section curvature $C = \frac{\varepsilon}{r^2}$ with $\varepsilon = \pm 1$ depending on the kind of the space. Then the following holds:*

$$F_{ij} = \frac{4\varepsilon\pi}{3r^2}\ \nabla_i J_j. \tag{4}$$

Proof. If (M, g) is a de Sitter space-time then the covariant curvature tensor R has the following components

$$R_{ijkl} = \frac{\varepsilon}{r^2}\ (g_{ik}g_{jl} - g_{il}g_{jk}). \tag{5}$$

Using (1), (2) and (5) we obtain (4). □

Proposition 5. *The energy-momentum tensor of the electromagnetic field is a conformal Killing symmetric tensor.*

Proof. The energy-momentum tensor of the electromagnetic field has the local components (see [1])

$$T_{ij} = \frac{1}{4\pi}\ (F_{ik}F_j^k - \frac{1}{4}\ F_{kl}F^{kl}\ g_{ij}).$$

We can see that the energy-energy tensor T is the trace-free part of the tensor field

$$G_{ij} = T_{ij} + \frac{\pi}{8}\ F_{kl}F^{kl}\ g_{ij}.$$

A symmetric covariant tensor field K on (M, g) is said to be Killing (see [4]) if the completely symmetric part of ∇K vanishes. Then, returning to pre-Maxwell equations (1), we can write $\nabla_i G_{jk} + \nabla_j G_{ki} + \nabla_k G_{ij} = 0$. Now, using the preceding definition, we obtain that G is a Killing tensor field (see [4]). A trace-free symmetric tensor field G on (M, g) is said to be conformal Killing (see [4]) therefore the energy-momentum tensor T is a conformal Killing tensor. □

If η is the skew-symmetric Levi-Civita pseudo-tensor (see [4]) with components η_{ijkl} then we obtain the following property of the electromagnetic field tensor F.

Proposition 6. *The tensor Y with components $Y_{kl} = \eta^{ij}{}_{kl}F_{ij}$ is a Killing-Yano tensor.*

Proof. Operating ∇_k to $Y_{kl} = \eta^{ij}{}_{kl}F_{ij}$ we get

$$\begin{aligned} \nabla_m Y_{kl} &= \nabla_m(\eta^{ij}{}_{kl}F_{ij}) = \eta^{ij}{}_{kl}\nabla_m F_{ij} \\ &= \frac{4\pi}{3}\ \eta^{ij}{}_{kl}(J_i g_{mj} - J_j g_{mi}) = \frac{8\pi}{3}\ \eta_{imkl}J^i. \end{aligned} \tag{6}$$

A Killing-Yano tensor is a skew-symmetric tensor Y with components Y_{ij} satisfying the Killing-Yano equations $\nabla_k Y_{ij} + \nabla_i Y_{kj} = 0$ (see [4]). The equations (6) show that Y with components $Y_{kl} = \eta^{ij}{}_{kl}F_{ij}$ is a Killing-Yano tensor. □

3. Integrals of the pre-Maxwell equations in a flat space-time

We will consider the pre-Maxwell equations (1.1) with the Ricci identities (2.2) as a closed system of differential equations of Cauchy type on (M, g) where F_{ij} and J_i are local components of unknown the tensor F and vector J (see [5]). For initial Cauchy conditions $F_{ij}(p_0) = F^0_{ij}$ and $J_i(p_0) = J^0_i$, where $p_0 \in M$, the system of equations (1.1) and (2.2) has at most one solution. Moreover, the general solution of the system of equations (1.1) and (2.2) depends no more on 8 real parameters.

Let us consider a flat space-time with a Lorentzian coordinate system $\{x^0, x^1, x^2, x^3\}$. It is clear that equations (3) may be rewritten as $\partial_j J_i$ for all $i, j = 0, 1, 2, 3$. Then the integrals of the pre-Maxwell equations (1) will take the form

$$F_{ij} = \frac{4\pi}{3}\,(C_i x^j - C_j x^i + C_{ij}), \tag{7}$$

where C_i are constant components of the current 4-vector J and C_{ij} are constant components of arbitrary skew-symmetric 2-tensor. It means the following: For a flat space-time (M, g) the system of equations (1.1) and (2.2) is complete integrable system and the number of real parameters of the general solution of the system reaches its maximum.

Using the classical notation (see [1]) we introduce the electric field intensity vector $\mathbf{E}$ and magnetic induction vector $\mathbf{B}$:

$$\mathbf{E} = \{F_{10}, F_{20}, F_{30}\} \quad \text{and} \quad \mathbf{B} = \{F_{23}, F_{31}, F_{21}\}.$$

Then from formulas above we obtain

$$\partial\mathbf{E}/\partial t = \{-C_1, -C_2, -C_3\} \quad \text{and} \quad \partial\mathbf{B}/\partial t = \{0, 0, 0\}$$

where x^0 is interpreted as the time t.

4. Integrals of the pre-Maxwell Equations in a de Sitter Space-Time

Let (M, g) be a de Sitter space-time. Then (see [6]) there exists a local coordinate system $\{x^0, x^1, x^2, x^3\}$ in which an arbitrary Killing-Yano 2-tensor Y has components $Y_{ij} = \mathrm{e}^{3\psi}\,(A_{mij}x^m + B_{ij})$ where $\psi = \frac{1}{10}\ln|\det g|$ and A_{mij}, B_{ij} are components of arbitrary two constants skew-symmetric tensors on (M, g).

On the other hand, an arbitrary closed conformal Killing 2-tensor F has following local components $F_{kl} = \eta^{ij}{}_{kl}\,Y_{ij}$ for components Y_{ij} of some Killing-Yano 2-tensor Y (see [6], [7]). Therefore the integrals of the pre-Maxwell equations (1.1) have the form $F_{kl} = *Y_{kl} = \eta^{ij}{}_{kl}Y_{ij}$ which implies $F_{ij} = \mathrm{e}^{3\psi}(\bar{A}_{mkl}x^m + \bar{B}_{kl})$ where $\bar{A}_{mkl} = \eta^{ij}{}_{kl}A_{mij}$ and $\bar{B}_{kl} = \eta^{ij}{}_{kl}B_{ij}$ (see the remark in the first paragraph).

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Serret's Curves, their Generalization and Explicit Parametrizations

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Peter A. Djondjorov and Vassil M. Vassilev

Abstract. Here we apply our original scheme for the reconstruction of plane curves whose curvatures are specified by functions of the radial coordinate to the curves introduced by J.-A. Serret. These curves are associated with the natural numbers and we extend their definition in order to include them into a family of curves depending on two continuous real parameters. The explicit parametrization of this new class of curves is presented as well.

Mathematics Subject Classification (2010). Primary 53A04; Secondary 53A55, 53A17.

Keywords. Classical differential geometry, plane curves, curvature, Frenet-Serret equations.

1. Introduction

Long time ago Serret [1] has described a family of plane algebraic curves in response to a question raised by Legendre. The problem was to find algebraic curves other than the lemniscate, such that their arc lengths are expressed by elliptic integrals of the first kind, and Serret claimed that he has found all such rational curves. Besides he provides a mechanical procedure [2] for their construction which will be described in the next Section. Before that we will mention that the original Serret curves were indexed by natural numbers but Liouville [3] had recognized immediately that rational numbers are suited as well as they also lead to algebraic curves. This has been further elucidated in Krohs' dissertation [4]. Here, we extend the definition of Serret's curves from discrete to continuous two-parameter family and present their explicit parametrizations.

Actually, the organization of the paper is as follows. The next section presents the mechanical construction of Serret's curves followed by another one in which the Frenet-Serret equations are formulated in Cartan moving frame. Then we outline the general scheme for the reconstruction of the plane curves whose curvature is given as a function of the radial coordinate and exemplify it by deriving the explicit parametrization of the Serret's curves.

2. Serret's curves

These curves were introduced as a trace of the end point M of the plane linkage bar shown in Figure 1. The lengths of the hinged rods OP and PM are specified

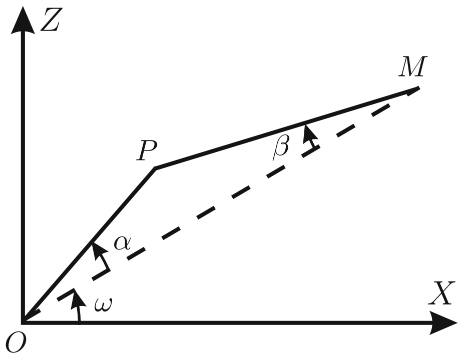

FIGURE 1. Serret's construction.

by a natural number $n \in \mathbb{N}$ via the formulas $\sqrt{n}$ and $\sqrt{n+1}$ respectively, the point O is fixed at the origin of the Cartesian coordinate system XOZ and the point M describing the curve $\mathcal{S}_n$ moves according to the rule

$$\cos\omega = \cos(n\alpha - (n+1)\beta), \tag{1}$$

where the angles α, β and ω are shown in Figure 1. Their analytical treatment is based on the application of the cosine theorem to the triangle OMP which gives

$$\cos\alpha = \frac{r^2 - 1}{2\sqrt{n}r}, \qquad \cos\beta = \frac{r^2 + 1}{2\sqrt{n+1}r}\cdot \tag{2}$$

Following Liouville's observation we obtain an algebraic curve even after replacing the index n with the rational number $\nu = p/q$, where $p, q \in \mathbb{N}$. This can be seen to be true by expressing the right-hand side of (1) as a polynomial in $\sin\alpha, \cos\alpha, \sin\beta$ and $\cos\beta$ and then by making use of the geometrical relations (2) to obtain the algebraic relations between x and r and z and r in the form of polynomial equations, i.e.,

$$P(x, r) = 0, \qquad Q(z, r) = 0. \tag{3}$$

Eliminating r between them one ends with some polynomial relation

$$F(x, z) = 0 \tag{4}$$

and this proves that the curve $\mathcal{S}_\nu$ is algebraic.

We should note that recently Lipkovski [5] was able to prove that all Serret's curves $\mathcal{S}_n$ are rational ones, i.e., they admit rational parametrizations.

Going back to the original Serret's writings one can find in [2] a formula for the curvature of the curve $\mathcal{S}_n$, namely

$$\kappa(r) = \frac{3r}{2\sqrt{n(n+1)}} - \frac{2n+1}{2\sqrt{n(n+1)}r} \tag{5}$$

which depends solely on the radial coordinate r.

As we shall see this formula is a crucial one for the present paper but before that in the next two Sections we will review shortly the geometry of the plane curves and the scheme for reconstruction of the curve from its curvature.

3. The Frenet-Serret equations

If $x(s)$, $z(s)$ and $\theta(s)$ denote the Cartesian coordinates of a curve in the plane XOZ and the slope of the tangent to it with respect to the OX axis regarded as functions of the arc-length parameter s one has the following geometrical relations

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos\theta(s), \qquad \frac{\mathrm{d}z(s)}{\mathrm{d}s} = \sin\theta(s) \tag{6}$$

which can be deduced from the Frenet-Serret equations (see [6] and [7])

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{T}, \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} \tag{7}$$

as well (see also Figure 2) in which $\mathbf{x} = (x, z)$, $\mathbf{T}$ and $\mathbf{N}$ are respectively the position, the tangent and the normal vectors to the curve.

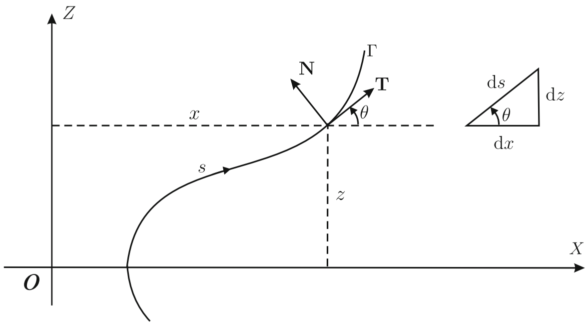

FIGURE 2. Geometry of the plane curve.

Let us recall that

$$\mathbf{T} = (\cos\theta, \sin\theta), \qquad \mathbf{N} = (-\sin\theta, \cos\theta). \tag{8}$$

We will proceed by going to the so-called co-moving frame $(\mathbf{T}, \mathbf{N})$ associated with the curve in which

$$\mathbf{x} = \xi\mathbf{T} + \eta\mathbf{N}. \tag{9}$$

According to the Frenet-Serret equations (7), the components $\xi(s)$ and $\eta(s)$ of the position vector with respect to the co-moving frame obey to the following equations

$$\dot{\xi} = \kappa\eta + 1, \qquad \dot{\eta} = -\kappa\xi. \tag{10}$$

Hereafter, the dot indicates a differentiation with respect to s. Below, it will be shown that in all cases in which the curvature of the plane curve depends solely on the distance from the origin, the above equations lead to explicit expressions (up to quadratures) for the slope angle θ and the position vector $\mathbf{x}$.

4. Reconstruction of the plane curve from its curvature

There are several ways for performing this task and for alternatives and illustrations the reader can see [8, 9, 10, 11, 12] and [13]. Here we will follow the scheme presented in [10] and start it by multiplying the first equation in (10) by ξ, the second one by η and summing the so obtained expressions in order to find

$$\xi = r\dot{r} \tag{11}$$

since one easily recognized that

$$\xi^2 + \eta^2 = r^2 \tag{12}$$

which follows from equation (9) and the very definition of the radial coordinate $r = |\mathbf{x}| = \sqrt{x^2 + z^2}$. Substituting the expression (11) back into the second of the equations (10) and integrating we obtain

$$\eta = m(r) + \mathring{c} \tag{13}$$

where

$$m(r) = -\int r\kappa(r)\mathrm{d}r \tag{14}$$

and $\mathring{c}$ is the integration constant. In view of equations (11) and (13), relation (12) leads to the following first-order ordinary differential equation

$$\dot{r}^2 = \frac{1}{r^2}\left[r^2 - (m(r) + \mathring{c})^2\right] \tag{15}$$

for the radial coordinate r.

Thus, given explicitly the curvature κ of a plane curve as a function of the radial coordinate r, one can try to express the general solution of equation (15) in a suitable analytical form. If such an attempt is successful, then the components ξ and η of the position vector can be found explicitly from equations (11), (13)–(15), while the expression for the slope angle θ can be obtained by solving the integral in the right-hand side of the relation

$$\theta(s) = \int \kappa(r(s))\mathrm{d}s \tag{16}$$

which is implied by the first of the equations (6).

In this way, the parametric equations according to formulas (8) and (9) can be written in the form

$$\begin{aligned} x(s) &= \xi(s)\cos\theta(s) - \eta(s)\sin\theta(s) \\ z(s) &= \xi(s)\sin\theta(s) + \eta(s)\cos\theta(s) \end{aligned} \tag{17}$$

in which all necessary ingredients are specified (up to integrations) from equations (11), (13) and (16).

5. Generalized Serret's curves and their explicit parametrization

The expression for the curvature of Serret's curves (5) suggests immediately a generalization of the form

$$\kappa(r) = 3\lambda r - \frac{\sigma}{r}, \qquad \lambda > 0, \qquad \sigma > 1. \tag{18}$$

Substitution of (18) in (13) produces

$$\eta = -\lambda r^3 + \sigma r \tag{19}$$

but one has to notice that the integration constant in (13) is taken to be zero. In these circumstances the differential equation (15) reduces to the equation

$$\frac{\mathrm{d}r}{\sqrt{(a^2 - r^2)(r^2 - c^2)}} = \lambda \mathrm{d}s \tag{20}$$

in which the real parameters a and c are given by the formulas

$$a = \sqrt{\frac{\sigma + 1}{\lambda}}, \qquad c = \sqrt{\frac{\sigma - 1}{\lambda}}\cdot \tag{21}$$

The integration of (20) can be performed in terms of the Jacobian elliptic function $\mathrm{dn}(\cdot,\cdot)$, namely

$$r(s) = a\,\mathrm{dn}(a\lambda s, k), \qquad k = \sqrt{\frac{2}{\sigma + 1}} \tag{22}$$

in which the first slot is occupied by its argument and the second one by the so-called elliptic modulus. More about elliptic functions and integrals can be found in [14] and [15].

The next step in the scheme amounts to the evaluation of the integral in (16) and this gives

$$\theta(s) = 3\mathrm{am}(\sqrt{\lambda(\sigma+1)}\,s, k) - \frac{\sigma}{\sqrt{\sigma^2 - 1}} \arccos \frac{\mathrm{cn}(\sqrt{\lambda(\sigma+1)}\;s, k)}{\mathrm{dn}(\sqrt{\lambda(\sigma+1)}\;s, k)} \tag{23}$$

where $\mathrm{am}(t, k)$ is the Jacobian amplitude function and $\mathrm{cn}(t, k) = \cos \mathrm{am}(t, k)$.

Having at disposal (11), (19), (22) and (23) one can enter into (17) and this gives the parametrization of the generalized Serret's curves. Obviously the parametrization of the classical Serret's curves can be obtained by taking

$$\lambda = \frac{1}{2\sqrt{n(n+1)}}, \qquad \sigma = \frac{2n+1}{2\sqrt{n(n+1)}}, \qquad n \in \mathbb{N} \tag{24}$$

and in this case the slope angle turns out to be

$$\theta_n(s) = 3\mathrm{am}(\mu_n s, k_n) - (2n+1)\arccos\frac{\mathrm{cn}(\mu_n s, k_n)}{\mathrm{dn}(\mu_n s, k_n)} \tag{25}$$

where

$$\mu_n = \frac{1}{2}\sqrt{\frac{2\sqrt{n(n+1)}+2n+1}{n(n+1)}}, \qquad k_n = 2\sqrt{\frac{\sqrt{n(n+1)}}{2\sqrt{n(n+1)}+2n+1}}\cdot \tag{26}$$

Several plots of both classical and generalized Serret's curves are presented in Figure 3 and Figure 4.

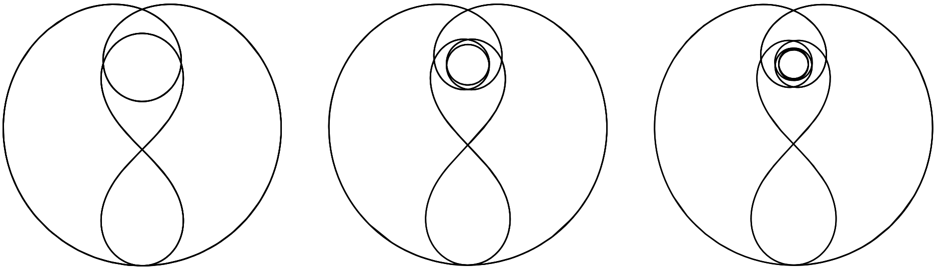

FIGURE 3. The classical Serret's curves $\mathcal{S}_1$ (left), $\mathcal{S}_2$ (middle) and $\mathcal{S}_3$ (right).

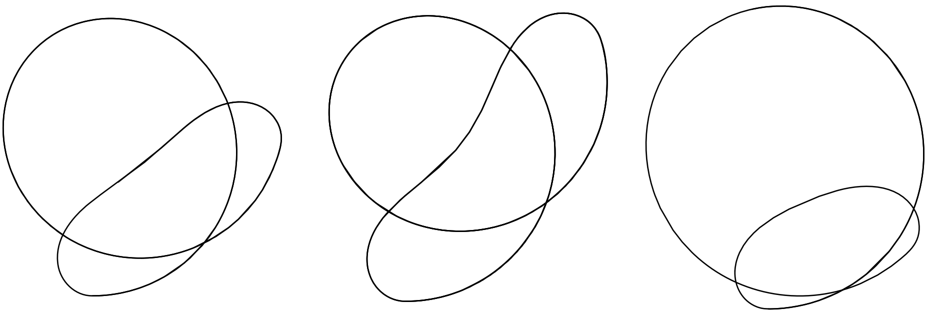

FIGURE 4. Three examples of the generalized Serret's curves $\mathcal{C}_1$ (left), $\mathcal{C}_2$ (middle) and $\mathcal{C}_3$ (right) generated respectively with parameter sets $\lambda = 1/3$, $\sigma = 7/5$, $\lambda = 4/3$, $\sigma = 9/7$ and $\lambda = 1/7$, $\sigma = 5/3$.

6. Concluding remarks

From the viewpoint of the curve engineering the curvature in (18) is a superposition of the Bernoulli's lemniscate [9] and Sturmian spiral [12]. On the other side Serret states that the curve $\mathcal{S}_1$ coincides with Bernoulli's lemniscate but looking at Figure 3 one can see that besides the lemniscate there exists an extra part of the curve. This discrepancy suggests also a more deep study of the whole family of Serret's curves and we hope to report on this subject elsewhere.

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Harmonic Spheres Conjecture

Armen Sergeev

Abstract. We discuss the harmonic spheres conjecture, relating the space of harmonic maps of the Riemann sphere into the loop space of a compact Lie group G with the moduli space of Yang–Mills G-fields on four-dimensional Euclidean space.

Mathematics Subject Classification (2010). Primary 58E20, 53C28,32L25.

Keywords. Harmonic spheres, Yang–Mills fields, instantons, loop spaces, Atiyah theorem, Donaldson theorem, Hilbert–Schmidt Grassmannian.

Introduction

There is a formal similarity between two classes of objects, arising in theoretical physics. These are harmonic maps of Riemann surfaces into Kähler manifolds (known in physics as classical solutions of sigma-model theory) and Yang–Mills fields. Both harmonic maps and Yang–Mills fields are critical points of some functionals – the energy functional in the case of harmonic maps and Yang–Mills action in the case of Yang–Mills fields. A similarity between these objects was noticed long ago by physicists but no mathematical explanation for this phenomena was known until in 1984 Atiyah found a relation between local minima of the above functionals. Namely, a theorem of Atiyah [1] establishes a 1–1 correspondence between the space of based holomorphic maps of the Riemann sphere $\mathbb{P}^1$ into the loop space ΩG of a compact Lie group G and the moduli space of G-instantons on four-dimensional Euclidean space $\mathbb{R}^4$. The harmonic spheres conjecture is obtained from this formulation by switching from local minima to critical points. Namely, it asserts that it should exist a natural 1–1 correspondence between the

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space of based harmonic maps of $\mathbb{P}^1$ into the loop space ΩG and the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$. In our paper we discuss this conjecture and present an idea of its proof.

The paper is organized in the following way. In Section 1 we introduce the harmonic spheres, i.e., harmonic maps of the Riemann sphere into Riemannian manifolds, starting with a simple, but instructive, example of harmonic maps from the Riemann sphere into itself. In Section 2 the Yang–Mills fields and instantons are defined in such a way which demonstrates explicitly a similarity between these objects and harmonic and holomorphic spheres respectively. The Atiyah theorem and harmonic spheres conjecture depend heavily on twistor interpretations of the introduced objects presented in Sections 3 and 4. In Section 3 we give a construction of the twistor bundle over $\mathbb{R}^4$ and formulate the theorems of Atiyah–Ward and Donaldson, yielding twistor interpretations of the moduli space of instantons on $\mathbb{R}^4$. In Section 4 a general definition of the twistor bundle over an arbitrary even-dimensional Riemannian manifold, due to Atiyah–Hitchin–Singer, is given together with the twistor interpretation of harmonic spheres, due to Eells–Salamon. As an application of the latter result we present in Section 5 the twistor interpretation of harmonic spheres in complex Grassmann manifolds. In Section 6 we switch to harmonic spheres in an infinite-dimensional Kähler manifold, namely, the loop space of a compact Lie group. We formulate the Atiyah theorem, mentioned above, and give an idea of its proof. The harmonic spheres conjecture is formulated in Section 7. In Section 9 we present an idea of its proof under additional assumptions, still to be checked.

1. Harmonic maps

Let us start from a simple model example arising in ferromagnetic theory. We consider a smooth map $\varphi : \mathbb{R}^2 \to S^2$ and define its *energy* by the Dirichlet integral

$$E(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} |d\varphi|^2 dx_1 dx_2.$$

We look for the maps φ with $E(\varphi) < \infty$ which are extremal with respect to the energy functional. Because of the finite energy condition it is natural to consider the maps stabilizing at infinity, i.e., $\varphi(x) \to \varphi_0$ for $|x| \to \infty$. Such maps extend continuously to the compactification $S^2 = \mathbb{R}^2 \cup \infty$ of $\mathbb{R}^2$.

The extended maps $\varphi : S^2 \to S^2$ have a topological invariant, called the *degree*, which may be computed by the formula

$$\deg \varphi = \int_{\mathbb{R}^2} \varphi^* \omega$$

where ω is the normalized volume form on S^2.

It is convenient to introduce the complex coordinate $z = x_1 + ix_2$ on the definition domain $\mathbb{R}^2$ of φ and stereographic complex coordinate w on the target

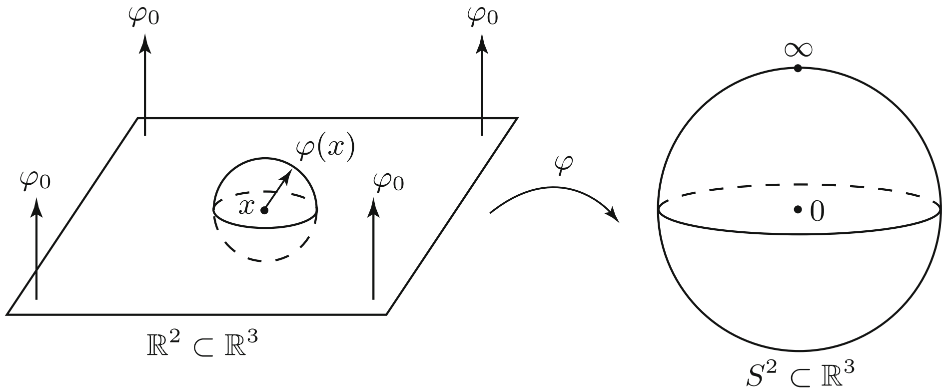


FIGURE 1

sphere $S^2 \setminus \{\infty\}$. The energy of the map $\varphi = w(z)$ in these coordinates will rewrite as

$$E(\varphi) = 2 \int_{\mathbb{C}} \frac{|\partial_z w|^2 + |\partial_{\bar{z}} w|^2}{(1 + |w|^2)^2} \, |dz \wedge d\bar{z}| \,,$$

while the degree of φ will be given by

$$\deg \varphi = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{|\partial_z w|^2 - |\partial_{\bar{z}} w|^2}{(1 + |w|^2)^2} \, |dz \wedge d\bar{z}| \,.$$

Comparing the last two formulas, we arrive at the inequality

$$E(\varphi) \geq 4\pi |\deg \varphi|.$$

The equality here is attained for deg $\varphi \geq 0$ only on holomorphic functions $\varphi = w(z)$, and for deg $\varphi < 0$ only on anti-holomorphic functions $\varphi = w(z)$. It follows that these functions realize local minima of the energy. To describe these minima more explicitly, we set the asymptotic value φ_0 equal to 1, using the SO(3)-invariance of the problem, and assume for definiteness that $k :=$ deg $\varphi > 0$. Then all minima of $E(\varphi)$ with fixed degree k will be given by rational functions of the form

$$\varphi = w(z) = \prod_{j=1}^{k} \frac{z - a_j}{z - b_j}$$

where $a_j \neq b_j$ are arbitrary complex numbers.

The smooth maps $\varphi : \mathbb{R}^2 \to S^2$ with $E(\varphi) < \infty$, which are critical points of the energy functional, will be called *harmonic*. It may be shown that in the considered case this functional has no other critical points apart from the described local minima. Identifying S^2 with the Riemann sphere $\mathbb{P}^1$, we can reformulate this result as follows: all harmonic maps $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ are given either by holomorphic or anti-holomorphic maps.

Generalizing this model example, we shall consider smooth maps $\varphi : \mathbb{P}^1 \to N$ from the Riemann sphere $\mathbb{P}^1$ into oriented Riemannian manifolds N.

Definition 1. A smooth map $\varphi : \mathbb{P}^1 \to N$ is called *harmonic* if it is a critical point of the energy functional given by the formula

$$E(\varphi) = \frac{1}{2} \int_{\mathbb{C}} |d\varphi|_N^2 \frac{|dz \wedge d\bar{z}|}{(1+|z|^2)^2}$$

where the modulus of differential $d\varphi$ is computed with respect to the Riemannian metric of N.

Harmonic maps $\varphi : \mathbb{P}^1 \to N$ will be called also *harmonic spheres* in N.

If the manifold N is Kähler, i.e., has a complex structure, compatible with the Riemannian metric, then holomorphic and anti-holomorphic maps $\varphi : \mathbb{P}^1 \to N$ will realize again local minima of the energy $E(\varphi)$. But, in contrast with the considered case $N = \mathbb{P}^1$, for Kähler manifolds of $\dim_{\mathbb{C}} N > 1$ there exist usually harmonic maps which are not locally minimal.

2. Instantons and Yang–Mills fields

Let G be a compact Lie group and A is a G-connection (*gauge potential*) on $\mathbb{R}^4$ given by the 1-form of type

$$A = \sum_{\mu=1}^{4} A_\mu(x) dx_\mu$$

with smooth coefficients $A_\mu(x)$, taking values in the Lie algebra $\mathfrak{g}$ of G. Denote by F_A the curvature of A (*gauge field*) given by the 2-form

$$F_A = \sum_{\mu,\nu=1}^{4} F_{\mu\nu}(x) dx_\mu \wedge dx_\nu$$

with coefficients, computed by the formula

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where $\partial_\mu := \partial/\partial x_\mu$, $\mu = 1, 2, 3, 4$, and $[\cdot, \cdot]$ denotes the commutator in the Lie algebra $\mathfrak{g}$.

Define the *Yang–Mills action* functional by the formula

$$S(A) = \frac{1}{2} \int_{\mathbb{R}^4} \mathrm{tr}(F_A \wedge *F_A)$$

where $*$ is the Hodge operator on $\mathbb{R}^4$, and the trace tr is computed with the help of a fixed invariant inner product on the Lie algebra $\mathfrak{g}$.

The functional $S(A)$ is invariant under *gauge transformations* given by

$$A \longmapsto A_g := g^{-1} dg + g^{-1} A g$$

where $g : \mathbb{R}^4 \to G$ is a smooth map, and G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation. It follows from the invariance of $S(A)$ under gauge transformations that the functional $S(A)$ depends in fact only on the class of connection A modulo gauge transformations.

Definition 2. Gauge fields with finite action $S(A) < \infty$, which are critical points of the functional $S(A)$, are called *Yang–Mills fields.*

Yang–Mills fields have an integer-valued topological invariant, called the *topological charge*, which is given by the formula

$$k(A) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}(F_A \wedge F_A).$$

If we write down the form F_A as

$$F_A = F_+ + F_-$$

with $F_\pm = \frac{1}{2}(*F_A \pm F_A)$ then the formulae for the action and charge will rewrite in the following form

$$S(A) = \frac{1}{2} \int_{\mathbb{R}^4} \left(\|F_+\|^2 + \|F_-\|^2\right) d^4x,$$

$$k(A) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \left(-\|F_+\|^2 + \|F_-\|^2\right) d^4x$$

where the norm $\|\cdot\|^2$ is computed with the help of invariant inner product on the Lie algebra $\mathfrak{g}$.

Comparing these two formulae, we arrive at the inequality

$$S(A) \geq 4\pi^2 |k(A)|.$$

Equality here is attained for $k > 0$ only on solutions of the equation

$$*F_A = -F_A, \tag{1}$$

and for $k < 0$ only on solutions of the equation

$$*F_A = F_A. \tag{2}$$

Definition 3. Solutions of equation (1) with finite action $S(A) < \infty$ are called *instantons*, and solutions of equation (2) with finite action $S(A) < \infty$ are called *anti-instantons.*

(Anti)instantons realize local minima of the action $S(A)$, however, there exist also non-minimal critical points of this functional.

Comparing harmonic maps with Yang-Mills fields, we notice immediately an evident analogy between:

$$\{\text{(anti)holomorphic maps}\} \longleftrightarrow \{\text{(anti)instantons}\}$$

and

$$\{\text{harmonic maps}\} \longleftrightarrow \{\text{Yang–Mills fields}\}.$$

We shall see later on that this evident analogy has, in fact, a deep meaning.

3. Twistor interpretation of instantons

We start from the construction of the twistor bundle over the Euclidean space $\mathbb{R}^4$ (cf. [2]). For that we compactify $\mathbb{R}^4$ to the Euclidean sphere $S^4 = \mathbb{R}^4 \cup \{\infty\}$ and identify S^4 with the quaternion projective line $\mathbb{HP}^1$. Points of $\mathbb{HP}^1$ are given by pairs $[z_1 + z_2 j, z_1' + z_2' j]$ of quaternions, not equal to zero simultaneously, which are defined up to the multiplication from the right by a nonzero quaternion.

The *twistor bundle* over $\mathbb{HP}^1$ has the form

$$\pi : \mathbb{P}^3 \xrightarrow{\mathbb{P}^1} \mathbb{HP}^1,$$

where $\mathbb{P}^3$ is the three-dimensional complex projective space, and may be considered as a complex analogue of the Hopf bundle

$$S^7 \xrightarrow{S^3} S^4.$$

It is defined by the tautological formula

$$[z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2 j, z_3 + z_4 j]$$

where the 4-tuple $[z_1, z_2, z_3, z_4]$ of complex numbers is defined up to multiplication by a nonzero complex number while the pair $[z_1 + z_2 j, z_3 + z_4 j]$ of quaternions is defined up to multiplication by a nonzero quaternion. The fibre of π coincides with the complex projective line $\mathbb{P}^1$.

The restriction of twistor bundle $\pi : \mathbb{P}^3 \to S^4$ to the Euclidean space $\mathbb{R}^4 = S^4 \setminus \infty$ yields the twistor bundle

$$\pi : \mathbb{P}^3 \setminus \mathbb{P}^1_\infty \longrightarrow \mathbb{R}^4 \tag{3}$$

where the eliminated projective line $\mathbb{P}^1_\infty$ coincides with the fibre of $\pi : \mathbb{P}^3 \to S^4$ at infinity.

According to Atiyah–Hitchin–Singer [3], the fibre of (3) at $x \in \mathbb{R}^4$ can be identified with the space of complex structures on the tangent space $T_x\mathbb{R}^4 \cong \mathbb{R}^4$, compatible with metric and orientation. Smooth sections of (3) may be considered, respectively, as almost complex structures on $\mathbb{R}^4$.

In terms of the twistor bundle $\pi : \mathbb{P}^3 \setminus \mathbb{P}^1 \to \mathbb{R}^4$ the *moduli space of G-instantons*, i.e., the quotient of the space of all G-instantons on $\mathbb{R}^4$ modulo gauge transformations, can be described by the following *Atiyah–Ward theorem*:

$$\left\{\begin{array}{l}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{(based) equivalence classes of holomor-}\\ \text{phic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^3\text{, holomorphi-}\\ \text{cally trivial on } \pi\text{-fibers}\end{array}\right\}.$$

Here, the term "based" means that the transformations, defining the equivalence of holomorphic $G^{\mathbb{C}}$-bundles over $\mathbb{P}^3$, should be identical on $\mathbb{P}^1_\infty$.

This result has the following two-dimensional reduction to the space $\mathbb{P}^1 \times \mathbb{P}^1$ given by *Donaldson theorem*:

$$\left\{\begin{array}{l}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{(based) equivalence classes of holomor-}\\ \text{phic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, holomor-}\\ \text{phically trivial on the union } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty\end{array}\right\}.$$

4. Twistor interpretation of harmonic spheres

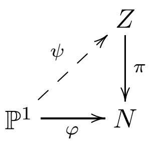
Using an interpretation of the twistor bundle $\mathbb{P}^3 \to S^4$, given in Section 3, we can define a *twistor bundle* over any even-dimensional oriented Riemannian manifold N. By definition, it is the bundle of complex structures on the manifold N, compatible with metric and orientation. In other words, $\pi : Z \to N$ is the bundle, associated with the bundle of oriented orthonormal frames on N, with fibre at $x \in N$ given by the space of complex structures J_x on the tangent space T_xN, compatible with metric and orientation. This fibre can be identified with the homogeneous space $\mathrm{SO}(2n)/\mathrm{U}(n)$ where $2n$ is the dimension of N. According to Atiyah–Hitchin–Singer [3], the twistor space Z can be provided with a natural almost complex structure, denoted by $\mathcal{J}^1$. This almost complex structure is integrable if the manifold N is conformally flat.

However, for the description of harmonic spheres in N we have to employ another almost complex structure which is defined in the following way. The Levi-Civita connection on N generates a connection on the twistor bundle $\pi : Z \to N$. In terms of this connection a new almost complex structure on Z, denoted by $\mathcal{J}^2$, is defined as

$$\mathcal{J}^2 = \begin{cases} -\mathcal{J}^1 & \text{along vertical } \pi\text{-directions} , \\ \mathcal{J}^1 & \text{along horizontal } \pi\text{-directions} . \end{cases}$$

This structure was introduced by Eells–Salamon [4] and is always non integrable. Harmonic spheres in N have the following interpretation in its terms.

Theorem 1 (Eells–Salamon [4]). *Projections* $\varphi = \pi \circ \psi$



of the maps $\psi : \mathbb{P}^1 \to Z$*, holomorphic with respect to the almost complex structure* $\mathcal{J}^2$*, are harmonic spheres in* N*.*

This theorem allows us to construct harmonic spheres in N from almost holomorphic spheres in the twistor space Z. So the original "real" problem of construction of harmonic spheres in the Riemannian manifold N is partially reduced to a "complex" problem of construction of holomorphic spheres in the almost complex manifold Z. It seems from the first glance that the latter problem is in no sense easier than the original one, especially taking into account that the almost complex structure $\mathcal{J}^2$ is never integrable. And there are examples of non-integrable almost complex structures which have no non-constant holomorphic functions even locally. However, we are dealing not with holomorphic functions $f : Z \to \mathbb{C}$ but rather with dual objects given by the maps $\psi : \mathbb{C} \to Z$, holomorphic with respect to the almost complex structure $\mathcal{J}^2$. Such maps are solutions of the $\bar{\partial}_J$-equation on $\mathbb{C}$ where $J := \psi^*(\mathcal{J}^2)$ is an almost complex structure on $\mathbb{C}$, induced by the map ψ

(which is integrable as any almost complex structure on a Riemann surface). In this way construction of holomorphic spheres in the space $(Z, \mathcal{J}^2)$ is reduced to the solution of a nonlinear Cauchy–Riemann equation on $\mathbb{C}$ with respect to the complex structure J. In particular, such an equation has many local solutions.

5. Harmonic spheres in complex Grassmann manifolds

We apply this twistor approach to the problem of construction of harmonic spheres in the complex Grassmann manifold $G_r(\mathbb{C}^d)$. Since $G_r(\mathbb{C}^d)$ is a homogeneous space of the unitary group $\mathrm{U}(d)$, it is natural to take for the twistor bundle in this case the bundle of complex structures on $G_r(\mathbb{C}^d)$ which are invariant under the action of $\mathrm{U}(d)$. Such a bundle coincides with a flag bundle over $G_r(\mathbb{C}^d)$ defined below.

Definition 4. The *flag manifold* $F_{\mathbf{r}}(\mathbb{C}^d)$ in $\mathbb{C}^d$ of *type* $\mathbf{r} = (r_1, \dots, r_n)$ with $d = r_1 + \cdots + r_n$ consists of the flags $\mathcal{W} = (W_1, \dots, W_n)$, i.e., nested sequences of complex subspaces

$$W_1 \subset \cdots \subset W_n = \mathbb{C}^d$$

such that the dimension of the subspace $V_1 := W_1$ is equal to r_1 and dimensions of the subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \le n$.

The flag manifold $F_{\mathbf{r}}(\mathbb{C}^d)$ admits the following description as a homogeneous space of the unitary group $\mathrm{U}(d)$:

$$F_{\mathbf{r}}(\mathbb{C}^d) = \mathrm{U}(d)/\,\mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n).$$

It is a compact complex manifold which has an $\mathrm{U}(d)$-invariant complex structure, denoted again by $\mathcal{J}^1$.

In order to construct the twistor flag bundle over the Grassmann manifold $G_r(\mathbb{C}^d)$ we fix an ordered subset $\sigma \subset \{1, \dots, n\}$ such that $\sum_{i \in \sigma} r_i = r$, and define the *flag bundle*

$$\pi_\sigma : F_{\mathbf{r}}(\mathbb{C}^d) \longrightarrow G_r(\mathbb{C}^d)$$

by setting

$$\pi_\sigma : \mathcal{W} = (W_1, \dots, W_n) \longmapsto W := \bigoplus_{i \in \sigma} V_i.$$

As in Section 4, we can provide the flag bundle π_σ with an almost complex structure $\mathcal{J}^2_\sigma$ so that the following analogue of Theorem 1 will hold.

Theorem 2 (Burstall–Salamon [5]). *The flag bundle*

$$\pi_\sigma : (F_{\mathbf{r}}(\mathbb{C}^d), \mathcal{J}^2_\sigma) \longrightarrow G_r(\mathbb{C}^d),$$

provided with an almost complex structure $\mathcal{J}^2_\sigma$, is a twistor bundle, i.e., the projection $\varphi = \pi_\sigma \circ \psi$ of any almost holomorphic sphere $\psi : \mathbb{P}^1 \to F_{\mathbf{r}}(\mathbb{C}^d)$ to $G_r(\mathbb{C}^d)$ is a harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ in $G_r(\mathbb{C}^d)$. Moreover, the converse assertion is also true: any harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(\mathbb{C}^d)$ in $G_r(\mathbb{C}^d)$ may be obtained in this way from some flag bundle $\pi_\sigma : F_{\mathbf{r}}(\mathbb{C}^d) \to G_r(\mathbb{C}^d)$.

So in this case the problem of construction of harmonic spheres in $G_r(\mathbb{C}^d)$ is completely reduced to the problem of construction of almost holomorphic spheres in flag bundles. Using this reduction, it was shown in [5] that any harmonic sphere in $G_r(\mathbb{C}^d)$ may be obtained by a Bäcklund-type transform combining holomorphic and anti-holomorphic spheres.

6. Atiyah theorem

We switch now to the case when our target manifold N is an infinite-dimensional Kähler manifold, namely the loop space of a compact Lie group.

Definition 5. The *loop space* of a compact Lie group G is

$$\Omega G := LG/G$$

where $LG = C^\infty(S^1, G)$ is the group of smooth loops in the group G and G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$.

The space ΩG is a Kähler Frechet manifold which has an LG-invariant complex structure. This structure is induced from the representation of ΩG as the quotient of the complex loop group $LG^{\mathbb{C}}$:

$$\Omega G = LG^{\mathbb{C}}/L_+G^{\mathbb{C}}$$

where $G^{\mathbb{C}}$ is the complexification of G and the subgroup $L_+G^{\mathbb{C}}$ consists of the loops $\gamma \in LG^{\mathbb{C}}$ which can be smoothly extended to holomorphic maps of the unit disc Δ into $G^{\mathbb{C}}$.

To formulate the Atiyah theorem, we recall the interpretation of the moduli space of G-instantons given by Donaldson theorem:

$$\left\{\begin{array}{l}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{(based) equivalence classes of holomor-}\\ \text{phic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, holomor-}\\ \text{phically trivial on the union } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty\end{array}\right\}.$$

The Atiyah theorem asserts that the right-hand side of this correspondence may be identified with the space of holomorphic spheres in ΩG. More precisely, there is a 1–1 correspondence between:

$$\left\{\begin{array}{l}\text{(based) equivalence classes of holomor-}\\ \text{phic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, holomor-}\\ \text{phically trivial on the union } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based holomorphic}\\ \text{spheres } f : \mathbb{P}^1 \to \Omega G,\\ \text{sending } \infty \text{ into the}\\ \text{origin of } \Omega G\end{array}\right\}.$$

The proof of Atiyah theorem is based on the following construction.

Consider the restriction of a holomorphic $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ to a projective line $\mathbb{P}^1_z$ which is parallel to P^1_∞ and goes through the point $\mathbb{P}^1 \times \{z\}$. It is determined by the transition function

$$\tilde{f}_z : S^1 \longrightarrow G^{\mathbb{C}}$$

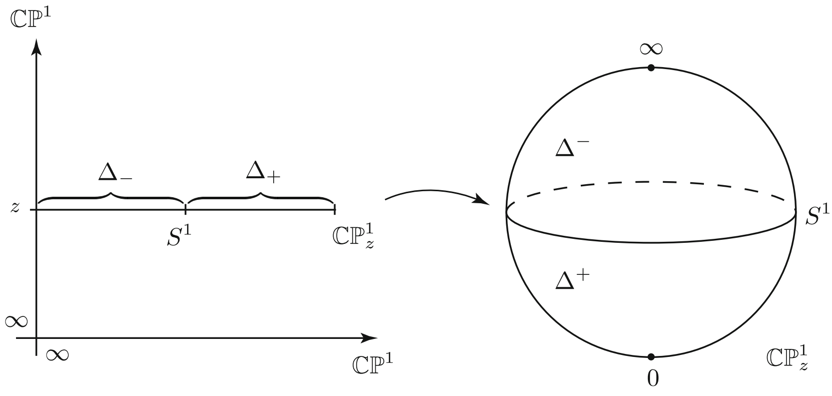


FIGURE 2

in the covering $\mathbb{P}^1_z = \overline{\Delta}_+ \cup \overline{\Delta}_-$ by lower and upper hemispheres of the sphere $\mathbb{P}^1_z$ and $\tilde{f}_z$ is holomorphic in a neighborhood of the equator $S^1 = \overline{\Delta}_+ \cap \overline{\Delta}_-$. Hence, $\tilde{f}_z \in LG^{\mathbb{C}}$ and we obtain a composite map

$$f : \mathbb{P}^1 \ni z \longmapsto \tilde{f}_z \in LG^{\mathbb{C}} \longmapsto f(z) \in \Omega G = LG^{\mathbb{C}}/L_+G^{\mathbb{C}}.$$

This map is holomorphic and based if and only if the original $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic and trivial on the union $\mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty$.

7. Harmonic spheres conjecture

The Donaldson and Atiyah theorems imply that there is a 1–1 correspondence between:

$$\left\{\begin{matrix}\text{moduli space of } G\text{-} \\ \text{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based holomorphic spheres} \\ f : \mathbb{P}^1 \to \Omega G\end{matrix}\right\} .$$

So we have a correspondence between local minima of two functionals, which were introduced before, namely

$$\left\{\begin{matrix}\text{Yang–Mills action, defined} \\ \text{on } G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \text{ and } \left\{\begin{matrix}\text{energy, defined on smooth} \\ \text{spheres in } \Omega G\end{matrix}\right\}$$

whose local minima are given respectively by

$$\left\{\text{(anti)instantons on } \mathbb{R}^4\right\} \text{ and } \{\text{(anti)holomorphic spheres in } \Omega G\} .$$

Replacing local minima by critical points of these functionals, we arrive at *harmonic spheres conjecture* asserting that it should exist a 1–1 correspondence between:

$$\left\{\begin{matrix}\text{moduli space of Yang–Mills} \\ G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based harmonic spheres} \\ \varphi : \mathbb{P}^1 \to \Omega G\end{matrix}\right\} .$$

This replacement of local minima by the critical points can be interpreted as a *"realification"* procedure. Indeed, if we replace smooth spheres in the right-hand side of the diagram by smooth functions $f : \mathbb{C} \to \mathbb{C}$ then the above procedure will reduce to the replacement of holomorphic and anti-holomorphic functions by arbitrary harmonic functions (which can be represented as sums of holomorphic and anti-holomorphic functions). In the case of smooth spheres in ΩG this switching from holomorphic and anti-holomorphic spheres to harmonic ones becomes non-trivial due to the non-linear character of Euler–Lagrange equations for the energy.

Unfortunately, a direct generalization of Atiyah–Donaldson proof to the harmonic case is not possible since the proof of Donaldson theorem, using the monad method, is purely holomorphic and does not extend directly to the harmonic case. However, one can try to reduce the proof of harmonic spheres conjecture to the holomorphic setting by "pulling-up" the both sides of the correspondence in the conjecture to the associated twistor spaces. The problem is that, while having a good description of the twistor space of harmonic spheres in ΩG (presented in the next Section), we do not know such a description for the moduli space of Yang–Mills fields on $\mathbb{R}^4$. So, apart from the proof of the harmonic spheres conjecture, we are also interested in obtaining the twistor description of this moduli space.

8. Twistor bundle over the loop space

For the construction of the twistor bundle over the loop space ΩG we first embed the space ΩG into an infinite-dimensional Grassmannian, and then construct the twistor bundle over this Grassmannian by analogy with the finite-dimensional case. The role of an infinite-dimensional Grassmannian will be played by the Hilbert–Schmidt Grassmannian of a complex Hilbert space.

Let H be a complex Hilbert space, having for its model the space $L^2_0(S^1, \mathbb{C})$ of square integrable functions on the circle with zero average. Suppose that H has a *polarization*, i.e., a decomposition

$$H = H_+ \oplus H_-$$

into the direct orthogonal sum of closed infinite-dimensional subspaces. In the case of $H = L^2_0(S^1, \mathbb{C})$ one can take for such subspaces

$$H_\pm = \{\gamma \in H : \gamma = \sum_{\pm k > 0} \gamma_k e^{ik\theta}\}.$$

Definition 6. The *Hilbert–Schmidt Grassmannian* $\mathrm{Gr}_{\mathrm{HS}}(H)$ consists of closed subspaces $W \subset H$ such that the orthogonal projection $\pi_+ : W \to H_+$ is Fredholm and the orthogonal projection $\pi_- : W \to H_-$ is Hilbert–Schmidt.

For a given subspace $W \in \mathrm{Gr}_{\mathrm{HS}}(H)$ the Fredholm index of the projection $\pi_+ : W \to H_+$ is called the *virtual dimension* of the subspace W.

The Hilbert–Schmidt Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ admits a homogeneous representation of the form

$$\mathrm{Gr}_{\mathrm{HS}}(H) = \frac{\mathrm{U}_{\mathrm{HS}}(H)}{\mathrm{U}(H_+) \times \mathrm{U}(H_-)}$$

where the *unitary Hilbert–Schmidt group* $\mathrm{U}_{\mathrm{HS}}(H)$ is

$$\mathrm{U}_{\mathrm{HS}}(H) = \{A \in \mathrm{U}(H) : \pi_- \circ A \circ \pi_+ \text{ is Hilbert–Schmidt}\}.$$

The Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ is a Hilbert Kähler manifold, consisting of a countable number of connected components of a fixed virtual dimension:

$$\mathrm{Gr}_{\mathrm{HS}}(H) = \bigcup_d G_d(H) \quad \text{where} \quad G_d(H) = \{W \in \mathrm{Gr}_{\mathrm{HS}}(H) : \text{virt.dim}\, W = d\}.$$

The virtual flag manifold $F^d_{\mathbf{r}}(H)$ is defined by analogy with the finite-dimensional case.

Definition 7. The *virtual flag manifold* $F^d_{\mathbf{r}}(H)$ in H of *type* $\mathbf{r} = (r_1, \ldots, r_n)$ with $d = r_1 + \cdots + r_n$ consists of flags $\mathcal{W} = (W_1, \ldots, W_n)$, i.e., nested sequences of complex subspaces

$$W_1 \subset \cdots \subset W_n \subset H$$

such that the virtual dimension of the subspace $V_1 := W_1$ is equal to r_1, and dimensions of subspaces $V_i := W_i \ominus W_{i-1}$ are equal to r_i for $1 < i \leq n$.

For the construction of the twistor flag bundle over the Grassmann manifold $G_r(H)$ we fix again an ordered subset $\sigma \subset \{1, \ldots, n\}$ such that $\sum_{i \in \sigma} r_i = r$, and define the *virtual flag bundle*

$$\pi_\sigma : F^d_{\mathbf{r}}(H) \longrightarrow G_r(H),$$

by setting

$$\pi_\sigma : \mathcal{W} = (W_1, \ldots, W_n) \longmapsto W := \bigoplus_{i \in \sigma} V_i.$$

As in the finite-dimensional case, the virtual flag bundle π_σ can be provided with an almost complex structure $\mathcal{J}^2_\sigma$ so that the following analogue of Theorem 2 will hold.

Theorem 3. *The virtual flag bundle*

$$\pi_\sigma : (F^d_{\mathbf{r}}(H), \mathcal{J}^2_\sigma) \longrightarrow G_r(H),$$

provided with the almost complex structure $\mathcal{J}^2_\sigma$, is a twistor bundle, i.e., the projection $\varphi = \pi_\sigma \circ \psi$ of any almost holomorphic sphere $\psi : \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ to $G_r(H)$ is a harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(H)$ in $G_r(H)$.

We think that the second part of Theorem 2, namely, the conversion of the above theorem is also true in this situation.

We construct now an isometric embedding of the loop space into a Hilbert–Schmidt Grassmannian. We suppose that the compact Lie group G is realized as a subgroup of the unitary group $\mathrm{U}(N)$ and construct an embedding of ΩG into the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ where we take for the Hilbert space H the space $L^2_0(S^1, \mathbb{C}^N)$.

We construct first an embedding of the loop group LG into the unitary Hilbert–Schmidt group $\mathrm{U}_{\mathrm{HS}}(H)$. For that we associate with a loop γ, belonging to the space $LG = C^\infty(S^1, G) \subset C^\infty(S^1, \mathrm{U}(N))$, a multiplication operator M_γ in the Hilbert space $H = L^2_0(S^1, \mathbb{C}^N)$, acting by the formula:

$$h \in H = L^2_0(S^1, \mathbb{C}^N) \longmapsto M_\gamma h(z) := \gamma(z)h(z),\ z \in S^1.$$

In other words, $M_\gamma h$ is a vector function from $H = L^2_0(S^1, \mathbb{C}^N)$, obtained by the pointwise application of the matrix function $\gamma \in C^\infty(S^1, \mathrm{U}(N))$ to the vector function $h \in H = L^2_0(S^1, \mathbb{C}^N)$. It is easy to check (cf. [6], Sec. 6.3) that the operator M_γ belongs to the unitary group $\mathrm{U}_{\mathrm{HS}}(H)$ if $\gamma \in C^\infty(S^1, \mathrm{U}(N))$.

The embedding $LG \hookrightarrow \mathrm{U}_{\mathrm{HS}}(H)$ generates an isometric embedding

$$\Omega G \longrightarrow \mathrm{Gr}_{\mathrm{HS}}(H).$$

9. Back to harmonic spheres conjecture

Using the isometric embedding $\Omega G \hookrightarrow \mathrm{Gr}_{\mathrm{HS}}(H)$, defined in the last section, we can consider a harmonic map $\varphi : \mathbb{P}^1 \to \Omega G$ as taking values in the Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$, hence, in one of the connected components $G_r(H)$ of the manifold $\mathrm{Gr}_{\mathrm{HS}}(H)$. To describe harmonic maps $\varphi : \mathbb{P}^1 \to G_r(H)$, we can proceed by analogy with the finite-dimensional case.

We formulate first the harmonic analogue of Atiyah theorem which asserts that there is a 1–1 correspondence between:

$$\left\{\begin{array}{l}\text{(based) equivalence classes of har-}\\ \text{monic } G^{\mathbb{C}}\text{-bundles over } \mathbb{P}^1 \times \mathbb{P}^1\text{, triv-}\\ \text{ial on the union } \mathbb{P}^1_\infty \cup \mathbb{P}^1_\infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based harmonic spheres}\\ f : \mathbb{P}^1 \to \Omega G\text{, sending } \infty\\ \text{into the origin}\end{array}\right\}.$$

To construct a harmonic $G^{\mathbb{C}}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, we proceed as in the holomorphic case. Suppose that a harmonic sphere $\varphi : \mathbb{P}^1 \to \Omega G \subset \mathrm{Gr}_{\mathrm{HS}}(H)$ is the projection of some harmonic sphere $\tilde\varphi : \mathbb{P}^1 \to LG^{\mathbb{C}}$ so that we have the following commutative diagram

$$\begin{array}{ccc} & & LG^{\mathbb{C}} \\ & \overset{\tilde\varphi}{\nearrow} & \big\downarrow \mathrm{pr} \\ \mathbb{P}^1 & \xrightarrow[\varphi]{} & \Omega G\,. \end{array}$$

Then $\tilde\varphi(z) \in LG^{\mathbb{C}}$ may be considered as the transition function for some harmonic bundle over $\mathbb{P}^1_z$ so that we have the following composite map

$$\{\varphi(z) \in \Omega G\} \longmapsto \{\tilde\varphi(z) \in LG^{\mathbb{C}}\} \longmapsto \left\{\begin{array}{l}\text{transition function of a}\\ \text{harmonic bundle over } \mathbb{P}^1_z\end{array}\right\}.$$

The obtained bundle over $\mathbb{P}^1_z$ is the restriction of a harmonic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, associated with the original harmonic map $\varphi : \mathbb{P}^1 \to \Omega G$.

In terms of Grassmannian $\mathrm{Gr}_{\mathrm{HS}}(H)$ the image $\varphi(z) \in \Omega G \subset \mathrm{Gr}_{\mathrm{HS}}(H)$ is identified with the subspace

$$W_z := M_{\tilde\varphi(z)} H_+$$

where M is the multiplication operator, introduced at the end of Section 7.

The *twistor interpretation* of this construction has the following form. A harmonic sphere in ΩG may be considered as a harmonic sphere in the Grassmannian $G_r(H) \subset \mathrm{Gr}_{\mathrm{HS}}(H)$, consisting of subspaces $W \subset H$ of some fixed virtual dimension r. Assume that the harmonic sphere $\varphi : \mathbb{P}^1 \to G_r(H)$ is the projection of some $\mathcal{J}^2_\sigma$-holomorphic sphere $\psi : \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ so that there is a commutative diagram

$$\begin{array}{ccc} & & F^d_{\mathbf{r}}(H) \\ & \overset{\psi}{\nearrow} & \big\downarrow \pi_\sigma \\ \mathbb{P}^1 & \xrightarrow[\varphi]{} & G_r(H)\ . \end{array}$$

The image $\psi(z) = (\psi_1(z), \dots, \psi_n(z))$ of a point $z \in \mathbb{P}^1$ under the map $\psi : \mathbb{P}^1 \to F^d_{\mathbf{r}}(H)$ is a virtual flag $\mathcal{W}_z = (W^1_z, \dots, W^n_z)$. Assume that every map $\psi_i : \mathbb{P}^1 \to G_{r_i}(H)$ is the projection of a map $\tilde{\psi}_i : \mathbb{P}^1 \to LG^{\mathbb{C}}$ so that

$$W^i_z = M_{\tilde{\psi}_i} H_+ .$$

Each of the maps $\tilde{\psi}_i$ can be considered as the transition function of some bundle over $\mathbb{P}^1_z$. It follows from the description of the almost complex structure $\mathcal{J}^2_\sigma$ on the twistor bundle π_σ that the maps $\tilde{\psi}_i$ determine either holomorphic, or anti-holomorphic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. So by Donaldson theorem such bundles correspond either to instantons, or anti-instantons on $\mathbb{R}^4$. This may be considered as a twistor construction of the moduli space of Yang–Mills fields on $\mathbb{R}^4$, associating with such a field a finite collection of instantons and anti-instantons on $\mathbb{R}^4$.

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Lax Equations and the Knizhnik–Zamolodchikov Connection

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Abstract. Given a Lax system of equations with the spectral parameter on a Riemann surface we construct a projective unitary representation of the Lie algebra of Hamiltonian vector fields by Knizhnik-Zamolodchikov operators. This provides a prequantization of the Lax system. The representation operators of Poisson commuting Hamiltonians of the Lax system projectively commute. If Hamiltonians depend only on the action variables then the corresponding operators commute

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1. Introduction

In [1] I. Krichever proposed a new notion of Lax operator with a spectral parameter on a Riemann surface. He has given a general and transparent treatment of Hamiltonian theory of the corresponding Lax equations. This work has called into being the notion of Lax operator algebras [2] and consequent generalization of the Krichever's approach on Lax operators taking values in the classical Lie algebras over $\mathbb{C}$ [3]. The corresponding class of Lax integrable systems contains Hitchin systems and their analog for pointed Riemann surfaces, integrable gyroscopes and similar examples.

In the present paper, given a Lax integrable system of the just mentioned type, we construct a unitary projective representation of the corresponding Lie

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algebra of Hamiltonian vector fields. For the Lax equations in question, we propose a way to represent Hamiltonian vector fields by covariant derivatives with respect to the Knizhnik-Zamolodchikov connection. It is conventional that Knizhnik-Zamolodchikov-Bernard operators provide a quantization of Calogero-Moser and Hitchin second-order Hamiltonians. Unexpectedly, we have observed such relation for all Hamiltonians, and, in a sense, for all observables of the Hamiltonian system given by the Lax equations in question.

The problem of a correspondence between an integrable system and a connection on a certain moduli space first has been addressed in the classical work [4] due to N. Hitchin. Two problems of N. Hitchin's approach are noted there: taking into account the marked points on Riemann surfaces and unitarity of the connection. It was pointed out that the Knizhnik-Zamolodchikov connection could be a solution of the first problem. As it is shown below, it resolves also the second one.

A large number of works is devoted to quantum integrable systems. Due to the space limitations we are not able to give all references. We restrict ourselves here with a (non complete) list of authors having contributed to the subject: B. Feigin and E. Frenkel, A. Beilinson and V. Drinfeld, A.P. Veselov, A.N. Sergeev, G. Felder, M.V. Feigin. The idea of quantization of Hitchin systems by means Knizhnik-Zamolodchikov connection was also addressed, or at least mentioned, many times in the theoretical physics literature (D. Ivanov, G. Felder and Ch. Wieczerkowski, M.A. Olshanetsky and A.M. Levin) but only the second-order Hamiltonians were involved.

The main results of the article are presented in Sections 3, 4. A review of the needed previous results is given in Section 2. We refer to [5] for more details, proofs, and missing references.

2. Phase space and Hamiltonians of a Lax integrable system

An integrable system considered here is given by the following data: a complex Riemann surface Σ, a classical Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, fixed points $P_1,\dots,P_N,P_\infty \in \Sigma$ ($N\in\mathbb{Z}_+$), a positive divisor $D=\sum\limits_{i=1}^{N} m_iP_i+m_\infty P_\infty$, points $\gamma_1,\dots,\gamma_K\in\Sigma$ ($K\in\mathbb{Z}_+$), and vectors $\alpha_1,\dots,\alpha_K\in\mathbb{C}^n$ associated with γ's. It is assumed that α's are given up to a common right action of the classical group G corresponding to $\mathfrak{g}$. The last two items (γ's and α's) are joined under the name *Tyurin data* [6].

2.1. Lax operators on Riemann surfaces

Let $\{\alpha\}=\{\alpha_i\}$, $\{\gamma\}=\{\gamma_i\}$, $\{\kappa\}=\{\kappa_i\in\mathbb{C}\}$, $\{\beta\}=\{\beta_i\in\mathbb{C}^n\}$, where $i=1,\dots,K$. Below, we avoid indices using α instead α_i, etc.

Consider a function $L(P,\{\alpha\},\{\beta\},\{\gamma\},\{\kappa\})$ ($P\in\Sigma$) such that L is meromorphic on Σ, has simple or double (depending on $\mathfrak{g}$) poles at γ's, may have poles at P_i's, is holomorphic elsewhere, and at every γ is of the form

$$L(z) = \frac{L_{-2}}{(z-z_\gamma)^2} + \frac{L_{-1}}{(z-z_\gamma)} + L_0 + L_1(z-z_\gamma) + O((z-z_\gamma)^2)$$

where z is a local coordinate at γ, $z_\gamma = z(\gamma)$, and the following relations hold:

$$L_{-2} = \nu\alpha\alpha^t\sigma, \quad L_{-1} = (\alpha\beta^t + \varepsilon\beta\alpha^t)\sigma, \quad \beta^t\sigma\alpha = 0, \quad L_0\alpha = \kappa\alpha \tag{1}$$

where α is associated with γ, β is arbitrary, $\nu \in \mathbb{C}$, σ is a $n \times n$ matrix. Besides, $\alpha^t\alpha = 0$ for $\mathfrak{g} = \mathfrak{so}(n)$, and $\alpha^t\sigma L_1\alpha = 0$ for $\mathfrak{g} = \mathfrak{sp}(2n)$. L is called a *Lax operator with a spectral parameter on the Riemann surface* Σ. The ν, ε, σ in (1) depend on $\mathfrak{g}$ as follows: $\nu \equiv 0$, $\varepsilon = 0$, $\sigma = id$ for $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$; $\nu \equiv 0$, $\varepsilon = -1$, $\sigma = id$ for $\mathfrak{g} = \mathfrak{so}(n)$; $\varepsilon = 1$, and σ is a matrix of the symplectic form for $\mathfrak{g} = \mathfrak{sp}(2n)$ for $\mathfrak{g} = \mathfrak{sp}(2n)$.

2.2. Lax operator algebras

Theorem 1 (Lie algebra structure, [2]). *For fixed Tyurin data the space of Lax operators is closed with respect to the point-wise commutator* $[L, L'](P) = [L(P), L'(P)]$ $(P \in \Sigma)$ (*in the case* $\mathfrak{g} = \mathfrak{gl}(n)$ *also with respect to the point-wise multiplication*).

It is called *Lax operator algebra* and denoted by $\overline{\mathfrak{g}}$.

Let $N = 1$, $\mathfrak{g}$ be simple. Then the following two theorems take place.

Theorem 2 (almost graded structure, [2]). *There exist subspaces* $\mathfrak{g}_m \subset \mathfrak{g}$ *such that*

$$(1)\ \overline{\mathfrak{g}} = \bigoplus_{\mathrm{m}=-\infty}^{\infty} \mathfrak{g}_{\mathrm{m}}; \quad (2)\ \dim \mathfrak{g}_{\mathrm{m}} = \dim \mathfrak{g}; \quad (3)\ [\mathfrak{g}_{\mathrm{k}}, \mathfrak{g}_{\mathrm{l}}] \subseteq \bigoplus_{\mathrm{m}=\mathrm{k}+\mathrm{l}}^{\mathrm{k}+\mathrm{l}+\mathrm{g}} \mathfrak{g}_{\mathrm{m}}.$$

Theorem 3. $\overline{\mathfrak{g}}$ *has only one almost graded central extension, up to equivalence* [7]. *It is given by the cocycle* $\gamma(L, L') = -\operatorname{res}_{P_\infty} \operatorname{tr}(LdL' - [L, L']\theta)$ *where* θ *is a certain* 1*-form* [2].

Theorem 2 and Theorem 3 hold true, with certain modifications, for a reductive $\mathfrak{g}$ (see [2] for $\mathfrak{g} = \mathfrak{gl}(n)$), and for $N > 1$ [8].

2.3. Lax equations

Let $M = M(z, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\kappa\})$ be defined by the same constrains as L, excluding $\beta^t\sigma\alpha = 0$, and $L_0\alpha = \kappa\alpha$, namely

$$M = \frac{M_{-2}}{(z-z_\gamma)^2} + \frac{M_{-1}}{z-z_\gamma} + M_0 + M_1(z-z_\gamma) + O((z-z_\gamma)^2)$$

where

$$M_{-2} = \lambda\alpha\alpha^t\sigma, \quad M_{-1} = (\alpha\mu^t + \varepsilon\mu\alpha^t)\sigma \tag{2}$$

M also takes values in $\mathfrak{g}$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}^n$.

For varying Tyurin data, let us consider the classical dynamics system having $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$, $\{\kappa\}$, and the main parts of L at $\{P_i | i = 1, \dots, N\}$ as dynamical variables. The equations of motion are given by the relation

$$\dot{L} = [L, M] \tag{3}$$

called *Lax equation*. In particular, the *equations of motion of Tyurin data* are as follows:

$$\dot{z}_\gamma = -\mu^t\sigma\alpha, \quad \dot{\alpha} = -M_0\alpha + \kappa\alpha.$$

For a positive divisor $D = \sum m_i P_i$ $(i = 1, \dots, N, \infty)$ such that supp D $\cap \{\gamma\} = \varnothing$, let $\mathcal{L}^D = \{L \in \bar{\mathfrak{g}} \mid (L) + D \geq 0$ outside γ's$\}$. Under a certain (effective) condition [1, 3] the Lax equation defines a flow on $\mathcal{L}^D$.

2.4. Examples

1) $g = 0$, $\alpha = 0$ (i.e., $\Sigma = \mathbb{C}P^1$, the bundle is trivial), $P_1 = 0$, $P_2 = \infty$. Then $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ is a loop algebra, (3) is a conventional Lax equation with rational spectral parameter:

$$L_t = [L, M], \quad L, M \in \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z), \quad z \in \mathbb{C}.$$

   The majority of known integrable cases of motion and hydrodynamics of a solid body belong to this class.
2) Elliptic curves: the above construction yields classical elliptic Calogero-Moser systems [3].
3) Arbitrary genus: for $D \in \mathcal{K}$, $\mathfrak{g} = \mathfrak{sl}(n)$ the construction gives the series A_n Hitchin system [1]. The similar should hold true for $\mathfrak{g} = \mathfrak{so}(n), \mathfrak{sp}(2n)$.

2.5. Hierarchy of commuting flows, and Hamiltonians

Theorem 4 ([1, 3]). *Given a generic L and effective divisor $D = \sum m_i P_i$ $(i = 1, \dots, N, \infty)$, there is a family of M-operators $M_a = M_a(L)$ $(a = (P_i, n, m)$, $n > 0$, $m > -m_i)$, unique up to normalization, such that outside the γ-points M_a has a pole at the point P_i only, and in the neighborhood of P_i*

$$M_a(w_i) = w_i^{-m} L^n(w_i) + O(1),$$

The equations $\partial_a L = [L, M_a]$ $(\partial_a = \partial/\partial t_a)$ define a family of commuting flows on an open subset of $\mathcal{L}^D$.

Given L, define matrices Ψ, K (where K is diagonal) by $\Psi L = K\Psi$. Let $\Omega = \mathrm{tr}(\delta\Psi \wedge \delta L \cdot \Psi^{-1} - \delta K \wedge \delta\Psi \cdot \Psi^{-1})$ where δ is the differential in α, β etc., ϖ be a holomorphic 1-form on Σ, and $2\omega = -\left(\sum \mathrm{res}_{\gamma_s} \Omega\varpi + \sum \mathrm{res}_{P_i} \Omega\varpi\right)$. Assume ϖ to be non-vanishing at the γ-points. Then ω *is a symplectic form on a certain closed invariant submanifold $\mathcal{P}^D \subset \mathcal{L}^D$* [1, 3].

Theorem 5 ([1, 3]). *The equations $\partial_a L = [L, M_a]$ are Hamiltonian with respect to the symplectic structure on $\mathcal{P}^D$ given by ω, and the Hamiltonians given by $H_a = -(n+1)^{-1} \mathrm{res}_{P_i}\, tr(w_i^{-m} L^{n+1}) dw_i$.*

Example (Further details of the example in Subsection 2.4). Let $\mathfrak{g} = \mathfrak{sl}(n)$, D be the divisor of ϖ. Then $\mathcal{L}^D \simeq T^*(\mathcal{M}_0)$ where $\mathcal{M}_0$ is an open subset of the moduli space of holomorphic vector bundles on Σ, H_a are Hitchin Hamiltonians.

3. Conformal field theory related to a Lax integrable system

By *conformal field theory* we mean a family of Riemann surfaces, a finite rank bundle (of *conformal blocks*) on this family, and a projectively flat connection (*Knizhnik-Zamolodchikov connection*) on this bundle. Given a Lax integrable system, we take the family of spectral curves over its phase space $\mathcal{P}^D$ for this purpose. In this section, following [9, 10], and references there, we prepare ingredients for the construction of the analog of Knizhnik-Zamolodchikov connection on this family.

3.1. Spectral curves, and the Kodaira–Spencer cocycle

For every $L \in \mathcal{P}^D$ (all arguments of L are fixed except for z) the curve Σ_L given by the equation $\det(L(z) - \lambda) = 0$ is called a *spectral curve* of L. It is a n-fold branch covering of Σ.

Given an arbitrary Riemann surface with marked points, the Lie algebra of meromorphic vector fields on it holomorphic outside the marked points is called *Krichever-Novikov vector field algebra.* Let $\mathcal{V}_L$ be the Krichever-Novikov vector field algebra on Σ_L, with the preimages of $P_1, \ldots, P_N, P_\infty$ as marked points. $\mathcal{V}_L$ is almost graded in the same sense as in Theorem 2.

Our next goal is to define a map $\rho : \mathrm{T}_L\mathcal{P}^D \to \mathcal{V}_L$. Fix a certain point, say $P_\infty \in \Sigma_L$, and think of P_∞ as of analytically depending on L. Choose a local family of transition functions d_L giving the complex structure on Σ_L and analytically depending on L. Let us take $X \in \mathrm{T}_L\mathcal{P}^D$ and a curve $L_X(t)$ in $\mathcal{P}^D$ with the initial point L and the tangent vector X at L. By definition

$$\rho(X) = d_L^{-1} \cdot \partial_X d_L. \tag{4}$$

We consider $\rho(X)$ as a local vector field on the Riemann surface Σ_L. Summarizing the results of [10, Sect. 5.1] we obtain

Proposition 6. *There exist $e \in \mathcal{V}_L$ such that in the neighborhood of P_∞ $\rho(X) = e$. The vector field e is defined modulo $\mathcal{V}_L^{(1)} \oplus \mathcal{V}_L^{\mathrm{reg}}$ where $\mathcal{V}_L^{(1)}$ is the sum of subspaces of non negative degree in $\mathcal{V}_L$, and $\mathcal{V}_L^{\mathrm{reg}} \subset \mathcal{V}_L$ consists of vector fields vanishing at P_∞. Both $\mathcal{V}_L^{(1)}$ and $\mathcal{V}_L^{\mathrm{reg}}$ are Lie subalgebras.*

Below, we always regard $\rho(X)$ as an element of $\mathcal{V}_L^{\mathrm{reg}} \backslash \mathcal{V}_L / \mathcal{V}_L^{(1)}$.

As a local vector field in the annulus centered at P_∞, $\rho(X)$ gives a certain Cech 1-cocycle of the Riemann surface Σ with coefficients in the tangent sheaf called *Kodaira-Spencer cocycle* of X. Its cohomology class is responsible for the deformation of moduli of the pointed surface along X.

3.2. Commutative Krichever–Novikov algebra, and its representation

Here, we canonically associate a commutative Krichever–Novikov algebra to a generic element $L \in \overline{\mathfrak{g}}$. We need it for the Sugawara construction below. Indeed, the Sugawara construction [11, 12, 13, 9] requires that the current algebra splits to the tensor product of a functional algebra and a finite-dimensional Lie algebra. Krichever–Novikov algebras are of this type, and Lax operator algebras are not.

Given L, let Ψ, K be as in Section 2.5. Ψ is defined modulo normalization and permutations of its rows (such normalization descends to the left multiplication Ψ by a diagonal matrix). By [3], in the neighborhood of a γ

$$\Psi(z) = \frac{\varepsilon\tilde{\beta}\alpha^t\sigma}{z - z_\gamma} + \Psi_0 + \cdots , \quad \Psi^{-1}(z) = \frac{\alpha\tilde{\beta}^t\sigma}{z - z_\gamma} + \tilde{\Psi}_0 + \cdots , \tag{5}$$

$$\Psi_0\alpha = 0, \ \varepsilon\alpha^t\sigma\tilde{\Psi}_0 = 0. \tag{6}$$

By [3, Lemma 7.4] K *is a meromorphic diagonal matrix-valued function on* Σ *holomorphic outside* P_i*'s* (i.e., a Krichever–Novikov matrix function). Conversely, let $\mathcal{A}$ be the Krichever–Novikov function algebra on Σ, and $\overline{\mathfrak{h}} = \mathfrak{h} \otimes \mathcal{A}$.

Lemma 7. *For any* $h \in \overline{\mathfrak{h}}$ *we have* $\Psi^{-1}h\Psi \in \overline{\mathfrak{g}}$.

Let $\mathcal{A}_L$ be the Krichever–Novikov function algebra on Σ_L having pre-images of the points P_i as the collection of poles. An arbitrary element of $\mathcal{A}_L$ can be pushed down to Σ as a diagonal matrix h. Every sheet is assigned with a certain row of h. The permutation ω of rows descends to the transformations $\Psi \to w\Psi$ (which is easily verified for w to be a transposition), and $h \to whw^{-1}$. Thus $L = \Psi^{-1}h\Psi$ is invariant, and we get a well-defined mapping $\mathcal{A}_L \to \overline{\mathfrak{g}}$.

By Lemma 7 any representation of $\overline{\mathfrak{g}}$ induces the corresponding representation of $\overline{\mathfrak{h}}$. Since Ψ is meromorphic at P_i's, the mapping $\overline{\mathfrak{h}} \to \overline{\mathfrak{g}}$ preserves degree. Hence an almost graded $\mathfrak{g}$-module induces the almost-graded $\overline{\mathfrak{h}}$-module.

Consider the following canonical representation of $\overline{\mathfrak{g}}$. Let $\mathcal{F}$ be the space of meromorphic vector-valued functions ψ holomorphic except at $P_1, \ldots, P_N$, P_∞, and γ's, such that $\psi(z) = \nu\alpha z^{-1} + \psi_0 + \cdots$ at any point γ. $\mathcal{F}$ is an almost graded $\overline{\mathfrak{g}}$-module with respect to the Krichever-Novikov base introduced in [14]. Consider the semi-infinite degree $\mathcal{F}^{\infty/2}$ of this module also constructed in [14]. The induced $\overline{\mathfrak{h}}$-module is what we need. This is an *admissible* module in the sense that every its element annihilates having been multiply operated by an element of $\overline{\mathfrak{h}}$ of a positive degree. Moreover, it is generated by a vacuum vector. By the above constructed mapping $\mathcal{A}_L \to \overline{\mathfrak{h}}$ we also consider $\mathcal{F}^{\infty/2}$ as an $\mathcal{A}_L$-module.

3.3. Sugawara representation

We present here the simplest (commutative) version of the Sugawara construction [12] (see also [13] for $N > 1$), in connection with $\mathcal{A}_L$.

Any admissible $\mathcal{A}_L$-module is equipped with a projective $\mathcal{V}_L$-action T canonically defined by the relation

$$[T(e), \pi(A)] = -c \cdot \pi(eA), \ A \in \mathcal{A}_L, \ e \in \mathcal{V}_L,$$

$\pi(A)$, $T(e)$ are the corresponding representation operators, eA denotes the natural action of a vector field on a function, c is the level of the $\mathcal{A}_L$-module From now on $V = \mathcal{F}^{\infty/2}$. For the effective definition of the representation T, more details, and generalizations see [13, 9, 10].

4. Representation of the algebra of Hamiltonian vector fields

Here we construct the Knizhnik-Zamolodchikov connection on the family of spectral curves. The Knizhnik-Zamolodchikov operators give a unitary projective representation of the Lie algebra of Hamiltonian vector fields. The behavior of the operators corresponding to the family of commuting Hamiltonians is investigated.

4.1. Conformal blocks and Knizhnik-Zamolodchikov connection

Let us consider the sheaf of $\mathcal{A}_L$-modules $\mathcal{F}^{\infty/2}$ on $\mathcal{P}^D$. Let $\mathfrak{h}^{reg} \subset \overline{\mathfrak{h}}$ be a subalgebra consisting of the functions regular at P_∞. The sheaf of quotients $\mathcal{F}^{\infty/2}/\mathfrak{h}^{reg}$ on $\mathcal{P}^D$ is called the sheaf of *covariants* (over a different base it was defined in [9] in this way).

Let X be a vector field on $\mathcal{P}^D$. By definition

$$\nabla_X = \partial_X + T(\rho(X)) \tag{7}$$

where ρ is the Kodaira-Spencer mapping, T is the Sugawara representation in $\mathcal{F}^{\infty/2}/\mathfrak{h}^{reg}$.

Theorem 8 ([9, 10]). *∇ is a projective flat connection on the sheaf of coinvariants:*

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + \lambda(X,Y)\cdot id$$

where λ is a certain cocycle, id is the identity operator.

We refer to ∇ as to *Knizhnik-Zamolodchikov* connection. The horizontal sections of the sheaf of covariants, with respect to ∇, are called *conformal blocks.*

4.2. Representation of Hamiltonian vector fields and commuting Hamiltonians. Unitarity

By Theorem 8 $X \to \nabla_X$ is a projective representation of the Lie algebra of vector fields on $\mathcal{P}^D$ in the space of sections of the sheaf of covariants. Denote this representation by ∇. The restriction of ∇ to the subalgebra of Hamiltonian vector fields gives the projective representation of that.

Theorem 9. *If X, Y are Hamiltonian vector fields such that their Hamiltonians Poisson commute then $[\nabla_X, \nabla_Y] = \lambda(X,Y)\cdot id$. If the Hamiltonians depend only on the action variables, then $[\nabla_X, \nabla_Y] = 0$.*

We refer to [5] for the proof of this theorem, as well as of Theorem 10.

Let $\mathcal{G}$ be a complex Lie algebra with an antilinear anti-involution $\dagger$, and T be its representation in the space V. An hermitian scalar product in V is called contravariant if $T(X)^\dagger = T(X^\dagger)$ where the $\dagger$ on the left-hand side means the hermitian conjugation. A pair consisting of T and a contravariant scalar product is called a unitary representation of $\mathcal{G}$ [11]. The restriction of T to the Lie subalgebra of the elements such that $X^\dagger = -X$ is unitary in the classical sense.

The Lie algebra of tangent vector fields on $\mathcal{P}^D$ belongs to the just defined class. Its antilinear anti-involution is pushed down from $\mathcal{V}_L$ with the help of the inverse to the Kodaira–Spencer mapping and the double-coset construction.

To construct a contravariant hermitian scalar product in the space of the representation ∇, first introduce a point-wise scalar product in the sheaf of covariants by declaring semi-infinite monomials with basis entries to be orthonormal ([11, p. 39], [12]). Then we integrate it over $\mathcal{P}^D$ by the volume form $\omega^p/p!$ which is invariant by the Poincaré theorem on absolute integral invariants of Hamiltonian phase flows.

Theorem 10. *The representation $\nabla : X \to \nabla_X$ of the Lie algebra of Hamiltonian vector fields on $\mathcal{P}^D$ in the subspace of smooth sections in $\mathcal{L}^2(C, \omega^p/p!)$ is unitary.*

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Short-time Asymptotics for Semigroups of Diffusion Type and Beyond

Stanislav A. Stepin

Abstract. In view of the asymptotic analysis to be carried out (for evolutionary semigroups beyond the diffusion type class) we first outline the path integral approach to the study of heat kernel asymptotics and heat trace estimations. Within this approach for the case of diffusion with a drift the heat kernel asymptotic properties are specified. Making use of parametrix expansion and Born approximation (instead of path integrals) we investigate semigroups generated by potential perturbations of bi-Laplacian: short-time asymptotics for the corresponding Schwartz kernel and regularized trace are derived.

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Keywords. Evolutionary semigroup, short-time asymptotics, parametrix expansion, Born approximation, regularized trace.

Wiener path integral representation for heat kernel

Let $U(t) = \exp(tH)$ be a Schrödinger semigroup generated by an operator

$$H \,=\, H_0 + V \,=\, \frac{1}{2}\,\Delta + V(x)\,,\ \ x \in \mathbb{R}^d\,,$$

with complex-valued potential V supposed to be bounded and continuous. The action of the semigroup is expressed by the Feynman-Kac formula (see, e.g., [1])

$$U(t)f(x) \,=\, \int_{\Omega_x} f(\omega(t))\exp\left(\int_0^t V(\omega(s))\,ds\right)\,d\mu_x(\omega),$$

where the integral with respect to Wiener measure μ_x is taken over the set $\Omega_x = \{\text{paths } \omega(s)\colon s \in [0,t], \omega(0) = x\}$.

The Feynman-Kac representation can be derived from the Duhamel equation

$$U(t) \,=\, U_0(t) \,+\, \int_0^t U_0(s)\,V\,U(t-s)ds\,,\ U_0(t) = \exp(tH_0)\,,$$

by iteration procedure and in fact it can be viewed as a summation formula for the perturbation theory series (Phillips-Dyson expansion)

$$U(t) \;=\; \sum_{n=0}^{\infty} U_n(t)\,, \;\; U_n(t) \;=\; \int_0^t U_0(s)\,V\,U_{n-1}(t-s)\,ds\,.$$

It follows that the Schrödinger semigroup integral kernel $p_V(x,y,t)$ (also known as a heat kernel) is given by the formula

$$p_V(x,y,t) = \int_{\Omega^t_{x,y}} \exp\left(\int_0^t V(\omega(s))ds\right) d\mu^t_{x,y}(\omega)\,. \tag{1}$$

The integration here is taken over the set $\Omega^t_{x,y}$ of paths $\omega(s)$ starting from point x and coming to y at time t with respect to conditional Wiener measure $\mu^t_{x,y}$ of the full mass $p_0(x,y,t)$.

The path integral representation enables one to derive short-time asymptotics of the heat kernel:

$$p_V(x,y,t) \;\sim\; p_0(x,y,t)\,\left\{1+\sum_{n=1}^{\infty} c_n(x,y)t^n\right\},\;\; t\downarrow 0\,.$$

To this end one should first expand the integrand in (1) into the time-power series and then carry out integration with respect to conditional Wiener measure. This approach gives explicit (i.e., non-recurrent) formulas for the coefficients $c_n(x,y)$ related to the so-called heat invariants. On the diagonal, heat kernel asymptotics takes the form

$$p_V(x,x,t) = (2\pi t)^{-3/2}\left\{1+tV(x)+\frac{t^2}{2}\left(\frac{1}{6}\Delta V(x)+V(x)^2\right)\right.$$
$$\left. +\frac{t^3}{6}\left(\frac{1}{40}\Delta^2 V(x)+\frac{1}{4}\langle\nabla V(x)\rangle^2+\frac{1}{2}V(x)\Delta V(x)+V(x)^3\right)+O(t^4)\right\}.$$

Coefficients here are homogeneous in the potential and its derivatives if we agree that each differentiation adds $1/2$ to the homogeneity degree of the corresponding summand (cf., [2]).

Formula (1) provides also an approach to estimating the regularized heat trace

$$\mathrm{Tr}(U(t)-U_0(t)) \;=\; \int (p_V(x,x,t)-p_0(x,x,t))\,dx$$

based on its path integral representation

$$\int dx \int_{\Omega^t_{x,x}} \left\{\exp\left(\int_0^t V(\omega(s))\,ds\right) - 1\right\}\, d\mu^t_{x,x}(\omega)\,,$$

application of convexity-type inequality and taking advantage of phase space bounds technique (cf., [1]). Such estimates are obtained in a similar way as their counterparts are derived within Berezin's Wick and anti-Wick symbolic calculus. To be compared with Theorem 4 below, the corresponding heat trace asymptotic estimate is presented here in the case of three-dimensional phase space (cf., [3]).

Theorem 1. *Given continuous bounded real-valued potential* $V \in \mathrm{L}_1(\mathbb{R}^3)$ *the difference* $U(t) - U_0(t)$ *is of trace class and the following inequalities*

$$(2\pi)^{-3/2} t^{-1/2} \int V(x)\, dx \leqslant \operatorname{Tr}(U(t) - U_0(t)) \leqslant (2\pi t)^{-3/2} \int \left(e^{tV(x)} - 1\right) dx$$

hold for arbitrary $t > 0$; *moreover, the short-time asymptotic formula is valid:*

$$\operatorname{Tr}(U(t) - U_0(t)) = (2\pi)^{-3/2} t^{-1/2} \int V(x)\, dx + O(\sqrt{t}).$$

As regards the upper bound of the regularized heat trace, it is known (see [4]) that for potentials $V(x)$ decaying sufficiently rapidly the estimate

$$|\operatorname{Tr}(U(t) - U_0(t))| \leqslant C(V)\, t^{-1/2}$$

holds provided H has purely continuous spectrum and point 0 is not a spectral singularity of H. Theorem 1 supplements this estimate and shows how sharp it is; note that the lower bound of the regularized trace turns out to be exact in the sense that it cannot be further improved due to the short-time asymptotics.

Two-sided estimates for the regularized trace of Schrödinger semigroup were derived in [5] by the application of a trace formula expressed in terms of the spectral shift function. One of the results obtained there is a simple corollary to the lower bound in Theorem 1, which was found by the author in [6] with the usage of path integral technique.

Diffusion with a drift: Feynman-Kac-Itô formula

The path integral approach proves to be useful in a rather general setting. Thus it does work in the case of diffusion with a drift when the semigroup generator is of the form

$$H = H_0 + A = \frac{1}{2}\Delta + \langle a(x)\nabla\rangle, \quad x \in \mathbb{R}^d.$$

The corresponding heat kernel $p_a(x, y, t)$ is then given by Feynman-Kac-Itô formula

$$p_a(x, y, t) = \int_{\Omega^t_{x,y}} \exp\left(\int_0^t \big\langle a(\omega(s))\, d\omega(s)\big\rangle - \frac{1}{2}\int_0^t a^2(\omega(s))\, ds \right) d\mu^t_{x,y}(\omega) \tag{2}$$

where the first summand in the exponent argument makes sense as an Itô stochastic integral.

This representation may be derived from an appropriately rearranged perturbation theory expansion of the semigroup e^{tH} obtained by iterations from the Duhamel equation

$$e^{tH} = e^{tH_0} + \int_0^t e^{sH_0} A\, e^{(t-s)H}\, ds.$$

For example, the second term $\int_0^t e^{sH_0} A\, e^{(t-s)H_0}\, ds$ in the corresponding iteration series has the integral kernel

$$\begin{aligned}
&\int_0^t ds \int p_0(x,\xi,s)\, \langle a(\xi)\nabla_\xi p_0(\xi,y,t-s)\rangle\, d\xi \\
&\quad = \int_0^t \int p_0(x,\xi,s)\, d\xi \int p_0(\xi,\eta,ds)\, \langle a(\xi)(\eta-\xi)\rangle\, p_0(\eta,y,t-(s+ds))\, d\eta \\
&\quad = \int_{\Omega^t_{x,y}} \left(\int_0^t \langle a(\omega(s))\, d\omega(s)\rangle \right) d\mu^t_{x,y}(\omega)\,.
\end{aligned}$$

The corresponding heat kernel $p_a(x,y,t)$ is known [7] to possess the asymptotics

$$p_a(x,y,t) \,\sim\, p_0(x,y,t)\, \exp\left(\int_0^1 \big\langle a(\xi(s))\, (y-x)\big\rangle\, ds \right),\ \ t \downarrow 0\,.$$

Integral representation (2) enables us to specify this formula. For the sake of simplicity let $d=3$ here.

Theorem 2. *Provided that drift coefficient* $a(x) \in \mathrm{C}^3(\mathbb{R}^3)$ *is bounded the following asymptotics holds*

$$\begin{aligned}
&p_a(x,y,t) \,=\, p_0(x,y,t)\, \exp\left(\int_0^1 \big\langle a(\eta(s))\, (y-x)\big\rangle\, ds \right) \\
&\times \Bigg\{ 1 \,+\, t \left(\frac{1}{2} \int_0^1 \big\langle \Delta a(\eta(s))(y-x)\big\rangle s(1-s) ds \,-\, \int_0^1 a^2(\eta(s))\, ds \right. \\
&\qquad - \int_0^1 \langle \nabla a\rangle(\eta(s))\, s\, ds \,+\, \int_0^1 (1-s) ds \int_0^s \Big[\big\langle \nabla\times a\,(\eta(s))\, \nabla\times a\,(\eta(r))\big\rangle\, (y-x)^2 \\
&\qquad \left. - \big\langle \nabla\times a\,(\eta(s))(y-x)\big\rangle \big\langle \nabla\times a\,(\eta(r))(y-x)\big\rangle \Big]\, r\, dr \right) \,+\, O(t^{5/4}) \Bigg\}
\end{aligned}$$

where $\eta(s) = x + (y-x)s$.

To outline the proof recall that conditional Wiener measure $\mu^t_{x,y}$ is supported (see [1]) on Brownian paths $\omega(s)$ starting from point x and coming to y at time t. Such paths are the trajectories of the process

$$\omega(s) \,=\, \eta(s/t) \,+\, \sqrt{t}\, b(s/t)$$

where $b(\tau)$ stands for three-dimensional Brownian bridge, i.e., Gaussian process with zero mean and covariance matrix

$$\mathrm{cov}\{b_i(\sigma), b_j(\tau)\} \,=\, (\min\{\sigma,\tau\} - \sigma\tau)\, \delta_{ij}\,.$$

Thus formula (2) can be rewritten in the form

$$p_a(x,y,t) = p_0(x,y,t)\,\mathbb{E}\left\{\exp\left(\int_0^1 \langle a(\eta(s)+\sqrt{t}b(s))(y-x)\rangle\,ds \right.\right.$$
$$\left.\left. + \sqrt{t}\int_0^1 \langle a(\eta(s)+\sqrt{t}b(s))\,db(s)\rangle - \frac{t}{2}\int_0^1 a^2(\eta(s)+\sqrt{t}b(s))\,ds\right)\right\}$$

where $\mathbb{E}$ denotes the expectation associated with Brownian bridge process. Now in order to extract short-time asymptotics one should expand the argument of the functional $\mathbb{E}$ into time-power series and then calculate expectations of its coefficients making use of stochastic integral formulas such as the following one (which is essentially due to Itô):

$$\mathbb{E}\left\{\int_0^1 \langle A(s)b(s)\,db(s)\rangle\right\} = -\int_0^1 \operatorname{Tr} A(s)\,sds\,.$$

Semigroups generated by perturbation of bi-Laplacian

Evolutionary semigroups which are beyond the diffusion type class will be considered now in the model setting when $U(t)=\exp(tH)$ is generated by

$$H = H_0 + V = -P(i\nabla) + V(x)$$

where $P(\xi) = |\xi|^4/4$; in fact bi-Laplacian H_0 may be replaced by an arbitrary elliptic operator with constant coefficients. Here we confine ourselves to three-dimensional case for the sake of simplicity.

The Schwartz kernel of the unperturbed semigroup $U_0(t)=\exp(tH_0)$ is given by the formula

$$G_0(x-y,t) = (2\pi)^{-3}\int \exp\big(-tP(\xi) + i\langle (x-y)\,\xi\rangle\big)\,d\xi \tag{3}$$

and admits the estimate

$$|G_0(x-y,t)| \leqslant (2\pi)^{-3}t^{-3/4}\exp\Big(-\frac{|x-y|^{4/3}}{4t^{1/3}}\Big)\int e^{-P(\xi)/10}\,d\xi\,.$$

The properties of the unperturbed semigroup integral kernel in a rather general situation when $P(\xi)$ is a positive definite form have been investigated in [8] and [9] (see also [10]). In order to study short-time asymptotic behavior of the kernel G_0 one can make use of the steepest descent (or saddle point) method applied to integral representation (3). It proves that as $t\downarrow 0$ the kernel G_0 is expressed by the asymptotic expansion

$$G_0(x-y,t) \sim \frac{2}{\sqrt{3}}\,\frac{(2\pi)^{-3/2}}{|x-y|\sqrt{t}}$$
$$\times\ \operatorname{Im}\left\{\exp\left(\frac{3}{4}\frac{|x-y|^{4/3}}{t^{1/3}}\,e^{-2\pi i/3}\right)\left(1 + \sum_{k=1}^{\infty} a_k\left(\frac{t^{1/3}}{|x-y|^{4/3}}\right)^k\right)\right\}.$$

Thus kernel $G_0(x-y,t)$ decays exponentially and oscillates as $t\downarrow 0$ so that (in contrast with the diffusion case) the kernel of the unperturbed semigroup $U_0(t)$ cannot be treated as a density of transition probability for a stochastic process.

In this situation the lack of path integral representation is supplemented by the parametrix expansion

$$G_V(x,y,t) \;=\; G_0(x-y,t) + \sum_{n=1}^{\infty} G^{(n)}(x,y,t) \tag{4}$$

which is just the perturbation theory series derived from the corresponding Duhamel equation by iteration procedure. The iterated kernels

$$G^{(n)}(x,y,t) \;=\; \int_0^t ds \int G_0(x-z,s)V(z)G^{(n-1)}(z,y,t-s)\,dz$$

for t sufficiently small (and a certain constant M large enough) admit the following estimates

$$|G^{(n)}(x,y,t)| \;\leqslant\; \frac{M^{n+1}}{\Gamma((n+1)/2)}\,t^{(n-1)/2}p(x,y,t)$$

which imply the asymptotics

$$\sum_{n>2} G^{(n)}(x,y,t) \;=\; O\big(t\,p(x,y,t)\big),\;\; p(x,y,t) \;=\; \exp\left(-\frac{3}{8}\frac{|x-y|^{4/3}}{t^{1/3}}\right).$$

A somewhat more delicate separate treatment of the second iterated kernel $G^{(2)}$ enables one to insert it into the estimate:

$$\sum_{n\geqslant 2} G^{(n)}(x,y,t) \;=\; O\big(t^{3/4}p(x,y,t)\big)\,.$$

Thus the principal (apart from G_0) term of the short-time asymptotics for the kernel G_V is determined by the first correction in the corresponding perturbation theory expansion (4), also known as Born approximation:

$$G^{(1)}(x,y,t) \;=\; \int_0^t ds \int G_0(x-z,s)V(z)G_0(z-y,t-s)\,dz\,.$$

Schwartz kernel short-time asymptotics

To deal with the Born approximation of the kernel G_V we will make use of quasi-probabilistic approach. Although the kernel G_0 is by no means a transition probability of a stochastic process some of its fundamental properties remain valid. For example the following mean value formula proves to be true

$$\int G_0(x-z,s)\,z\,G_0(z-y,t-s)\,dz \;=\; G_0(x-y,t)\,(x+(y-x)s/t)\,.$$

This relationship may be viewed as an analogue of mathematical expectation formula for the location (at an instant s) of the trajectory starting from x and coming to y at the instant t. Thus the Born approximation can be decomposed into the sum of the principal term and the mean deviation

$$\begin{aligned} G^{(1)}(x,y,t) &= G_0(x-y,t)\int_0^t V(\eta(s/t))\,ds \\ &\quad + \int_0^t ds \int G_0(x-z,s)\big(V(z)-V(\eta(s/t))\big)G_0(z-y,t-s)\,dz \\ &= t\,G_0(x-y,t)\int_0^1 V(\eta(\tau))\,d\tau \,+\, O\big(t^{7/12}p(x,y,t)\big) \end{aligned}$$

provided that $V(x)$ is twice continuously differentiable. Summing up we formulate

Theorem 3. *Let potential $V(x) \in \mathrm{C}^2(\mathbb{R}^3)\cap \mathrm{L}_1(\mathbb{R}^3)$ be bounded. Then an off-diagonal short-time asymptotics*

$$\begin{aligned} G_V(x,y,t) \;&=\; G_0(x-y,t)\bigg(1+t\int_0^1 V(x+(y-x)\tau)\,d\tau\bigg) \\ &\qquad + O\bigg(t^{7/12}\exp\bigg(-\frac{3}{8}\frac{|x-y|^{4/3}}{t^{1/3}}\bigg)\bigg) \end{aligned}$$

is valid where

$$\begin{aligned} G_0(x-y,t) &= \frac{2}{\sqrt{3}(2\pi)^{3/2}}\,\frac{t^{-1/2}}{|x-y|}\,\exp\bigg(-\frac{3}{8}\frac{|x-y|^{4/3}}{t^{1/3}}\bigg) \\ &\quad\times\bigg\{\sin\bigg(\frac{3\sqrt{3}}{8}\frac{|x-y|^{4/3}}{t^{1/3}}\bigg) + O\big(t^{1/3}\big)\bigg\}; \end{aligned}$$

besides, on the diagonal $x=y$ one has

$$G_V(x,x,t) \;=\; (2\pi)^{-3}t^{-3/4}\big(1+V(x)\big)\int e^{-P(\xi)}\,d\xi \,+\, O(\sqrt{t}),\; t\downarrow 0.$$

Short-time behavior of the regularized trace can be qualified under even weaker assumptions.

Theorem 4. *Given bounded potential $V(x)\in \mathrm{L}_1(\mathbb{R}^3)$ the difference $U(t)-U_0(t)$ is of trace class and*

$$\mathrm{Tr}\,\big(U(t)-U_0(t)\big) \;=\; t\,G_0(0,t)\int V(x)\,dx \,+\, O(\sqrt{t})\,,\; t\downarrow 0.$$

Validity of this formula (without any smoothness assumptions being imposed upon V) is due to the fact that the Born approximation integrated over the diagonal can be calculated explicitly:

$$\begin{aligned} \int G^{(1)}(x,x,t)\,dx \;&=\; \int dx\int_0^t ds\int G_0(x-z,s)V(z)G_0(z-x,t-s)\,dz \\ = \int_0^t ds\int V(z)\,dz\int G_0(x-z,s)&G_0(z-x,t-s)\,dx \;=\; t\,G_0(0,t)\int V(z)\,dz. \end{aligned}$$

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Bureaucratic World: Is it Unavoidable?

Bogdan Mielnik

Abstract. An excess and inefficiency of the control mechanisms in the present day societies is commented.

Esteemed Colleagues:

The remarks below concern a certain lack of equilibrium in the present day legislation, affecting the life and science, with rather adverse consequences for our work. The disequilibrium seems to privilege the fashionable problems such as the "political correctness", "sexual harassment, etc. While those are visibly exaggerated, some urgent subjects are left unattended. One of them is the

Bureaucratic Harassment,

an epidemic phenomenon, which grows without any reasonable limit.

Though the trouble is not new, its consequences in the present day society are increasingly awkward, causing serious doubts whether the democracy is indeed the best of the systems. The human life is affected by too many unnecessary and obviously absurd regulations which could be easily avoided by an enlightened medieval autocrat. (The whole problem is, of course, how to assure that the autocrat will be indeed enlightened!) Yet, we often feel that some of our problems would be solved in few minutes by a despotic ancient king, whereas they need some months or even years of struggles in our present day institutions.

The disease affects all areas, though it seems specially damaging for the activities which require some peace of mind, concentration and creative work. We refer, of course, to the arts and science. The damage to the science consists not only in our loss of time, but much more in the fact that the scientist of today is forced to subordinate himself to some counter-intellectual patterns of reports and planning, forcing him indeed to accept the professional dishonesty. The most absurd demand he faces is to present the program (and the time-table) of his future discoveries. Such plans can bring the best results if they fail, since only then they can reveal something new. In fact, the discoveries of radioactivity by Becquerel and by Pierre and Marie Curie, or penicillin by Alexander Fleming, occurred thanks to the frustrations of their initial projects. Neither the excursion of Christopher Columbus could accomplish his original plan to discover the shortest way to India.

The only thing discovered by CC was an obstacle, on which we live today!... When composing his irrelevant projects, ironically, the scientist is a victim of an almost paranoid suspicion, obliged to document every little detail of any routine spending, precisely when he intensely tries to be honest, at least in frames of the obligatory bureaucratic fiction!... (Needless to say, the truly significant frauds occur much above the bureaucratic control levels).

The abstract pollution...

More inconsistencies. While the urgent need to protect our natural environment is already recognized [1, 2, 3], the destruction of our lives by too many rules, documents, etc., that is, the *pollution of abstract environment,* progresses without any defense... The examples are abundant and increasingly alarming.

El Pais [4] describes the executives of one of the Town Councils increasing the bureaucratic demands – to enforce bribes for "resolving the problem"... The journal *Rzeczpospolita*, Poland, August 2011, reports a tragic error in an oncology clinic where the doctors removed the healthy kidney instead of the cancerous one. The journal comments: "the good specialists escape, but the administration grows". About a year ago, a bureaucratic homicide was committed in one of hospitals. A middle age man had a heart attack on the street. Somehow, he was still able to walk to the near hospital, but was not admitted because of the lack of obligatory documents. He died on the hospital steps. Unfortunately, such "incidents" are not exceptional... Meanwhile, the scientists cannot work, since they are too busy with bureaucratic plans and reports. The engineers cannot construct highways, since they are too busy navigating through the jungles of regulations. New forms of business appear: the enterprises which help the scientists to formulate their grant requests in terms convincing for the bureaucrats. (The corruptive consequences are not difficult to guess!...)

We think, you can easily provide a collection of your own examples. A question arises, how such phenomena could at all develop? To explain this, we formulated 4 laws of bureaucracy which you might find relevant:

Four Laws of Bureaucracy:

I. All attempts of the state administrations to improve the scientific work by bureaucratic projects, reports, etc. will be reduced to zero by the social organism – though not gratis: the price is an enormous increase of socially useless work.

II. What is the source of the incredible facility of public administrations in multiplying endlessly the prescriptions, formalities and obligatory documents? The reason is that the bureaucrats do not perform the bureaucratic work: they leave it to their victims.

III. In the bureaucratic environment the problems of little importance are always infinitely more urgent than the truly important ones. This is why *thou will never do anything important.*

IV. The knowledge of the four bureaucracy laws won't help you in anything.

Our formulations are deliberately simplified, just to illustrate the center of the problem. But... how can we break the law of impotence **IV**? Should we support the spontaneous rebellions? Do we dream to live in a complete anarchy? This, most evidently, cannot be the solution. The public administration must always exist. The only problem is to have a good administration instead of an excessive one. While the question is simple, the answer is not. The "bureaucratic disease" is a deep civilization crisis marking the "childhood's end" of humanity. Our distant and recent past shows how it developed.

From prehistory to the dark age

The prehistoric population obeyed shamans and tribal leaders. There was still no much bureaucracy. The subsequent evolution subordinated the human masses to the formal laws, assuring some stability in turbulent epochs. The industrial revolutions of the XVIII and XIX c. in Europe and America created new bureaucratic classes emerging from some (more or less) credible elections. It is interesting that the passports still did not exist at the beginning of XIX c.; they were invented by governments (supposedly) to facilitate travels. In XX c., some countries were affected by highly despotic types of bureaucracy. One, introduced in Russia by the communist party, was supposed to represent higher social formation, next after the capitalism, granting the social equality by eliminating the private property. To assure the universal equality, the soviet state was organized as a hierarchy of party levels, each higher supervising each lower one, with the corresponding administrative privileges. (But the majority of soviet citizens could not even dream about passports. The Soviet workers needed special permissions to travel inside of the Soviet state). Far from constituting a superior, post-capitalist society, the system developed only a highly unproductive economy, based on compulsive work in collective farms. In contrast with the neo-liberal ideas, it was indeed a neo-feudal society which could survive only due to an extreme bureaucratic control and terror, but finally collapsed, leaving in ruins one of the richest countries of the world. Equally disastrous results were achieved by the 'national socialism' in Germany with its ideology of hate and racial superiority. Both show how the brainwashed populations (including the scientists) can be dominated to the obsessive ideologies [5, 6], the danger which should not be forgotten.

Is the democracy failing?

When the totalitarian systems collapsed, it could seem that the best structure for modern society was at hand: it should follow the design of western democracies, with all its imperfections. This, of course, does not mean that all dreams of equality can be fulfilled. Since people are different, the "demos" cannot assure the equal status to everybody. It is enough to imagine an angry crowd marching and demanding: "One Mercedes Benz for each poor, one Mercedes Benz for each poor!"... to understand the impossibility of the truly egalitarian society. Even if the Mercedes factories had a sufficient production power, do you imagine the Earth surface devastated by billions of cars?

The democratic systems, henceforth, cannot achieve an authentic equality. At the best, they can be the "soft" versions of the "Brave New World" of Aldous Huxley [7]. The power should be in hands of enlightened elites. The rest of citizen (the *demos*) must have some decent jobs and salaries, but indeed, they are just a kind of biological reserve, which should live happy, without unnecessary ambitions... More precisely: to live in a harmless passivity (if not *idiocy?*), entertained by sport games, competitions of singers, etc., with some elementary education, sufficient to choose talented youngsters to renew the government, to the satisfaction of the *demos.*

Yet, even this design suffers some destructive mechanisms. One of them is a constant increase of bureaucracy, caused by the rapid growth of the human crowds, with inflationary phenomena visible in all areas of life, in particular, in science. The present day professionals are evaluated according to the number of publications rather than results. Under the bureaucratic pressure, this number blows up so fast, that the world journals have not enough experienced referees to evaluate the papers. So, the inexperienced authors must serve as the referees for their equally inexperienced colleagues. The editors are in trouble. They organize conferences asking to reduce the avalanche of publications, but the bureaucratically inflated bubble of "productivity" grows practically without any control. Recently, the editors of some journals took the task in their own hands, rejecting most of papers which do not seem to follow the promising trends, even though the method has some corruptive aspects (forming the privileged influence groups inside of the scientific community). Looking for some better criteria, the science administrators estimate the papers by their impact (citations). However, this too turns questionable (a lot of authors cite papers which they never read!). In the hasty, superficial development, the top achievements of the past become too difficult for the new generations and are left in oblivion (though from time to time rediscovered). In the bureaucratically organized science, many specialists feel that they will never advance if they won't occupy executive positions (and they are probably right!). The interdependence between the scientific and politico-bureaucratic levels extends everywhere. The well-known University rankings are based rather on public relations than on scientific status, and so are the titles of "Doctor *honoris causa*", offered usually to the politicians. So, did our civilization reached the top of its creative possibilities like the civilization of ants or termites? Perhaps not, but the future remains unclear.

Heavy or light pathologies?...

By trying to complete the picture, one cannot escape conclusion that the *bureaucracy* has no natural limits. Of course, apart of exceptions, the *bureaucrats* are not evil. They simply try to fulfill their work in peace, even if their peace destroys the peace of others. The results, though, are not at all innocent.

One of obsessive bureaucratic problems in European Union was the polemic about the legal definition of a carrot. Is the carrot a fruit or a vegetable? Of course, the question was motivated by financial problems which we skip here. The final verdict, after more than one decade of costly debates, was that the carrot is a fruit if cultivated in Portugal, otherwise it is a vegetable. Pleased by this success, the European bureaucracy invested the next efforts to define the legal parameters (size and curvature) of cucumbers and bananas… Worse, since the present day administrations try to apply the same method to define a good scientist: requesting the numbers of publications in prestigious journals, the numbers of graduated students, the list of financially supported "projects", etc. etc. … The detailed demands differ in various countries and institutions, but everywhere the scientists have a full time job reporting their numerical parameters. The phenomenon is not limited to the science. In almost all areas, the employees must report their parameters to demonstrate that they are enough productive (but not too conflictive!) to advance in hierarchies, to become directors, secretaries, government consultants, etc., etc. … (Poor human carrots?…).

Crisis and consequences

The situation is additionally complicated by the economical crisis, much deeper than the famous collapse of 1928. One of problems seems to be that the economy of the rich countries was too dependent on the redundant (unnecessary) goods. Only thanks to an enormous self-confidence (if not arrogance) of the consumers these products could be sold… Simultaneously, it turns out obvious that money is illusionary (even if the lack of money can be real!). The mega-frauds are so spectacular, that even pumping billions of illusionary money into illusionary goods might require some patience to save the situation… Worse, since the bureaucrats in panic try now to apply an inverse doctrine, by cutting funds for all areas, including the science (the famous "austerity"!), They also try to introduce some utilitarian principles into the scientific research. As turns out now, the scientists should not waste their time for abstract problems, but they should show their capacities by looking for innovations, patents, technological solutions, to improve the financial results of the decaying industries. The recipe, though, seems questionable:

1. The crisis started precisely in countries which were leaders in technology and innovation.
2. Around 1905 Albert Einstein was working in the Swiss patent office. Were he forced to dedicate his attention to invent patents, he might have no time and energy to write down his historical works on quantum theory of light and special relativity, with enormous looses for all patents of the future…
3. The innovations are not necessarily benign. The sequence of discoveries in the food conservation techniques permitted to achieve high profits in the industry of fast food and refreshments. However, the chemically conserved products are not neutral for the health of the consumers; they cause the overweight, diabetes, and many other troubles.
4. The modern industries are literally infested by innovations, so the examples can be multiplied at will. One of most typical situations is observed in medical industries which besides the impressive discoveries contains also a list of failures, from the well-known case of "*Thalomid*", up to the recent affair

of the plastic breast implants. (For other deceptions, see John le Carré, *Big Pharma,* Google.)

The present day crises might be indeed an occasion to invest (see Paul Krugman), but the investments should be the results of careful, long term projects, creating perdurable goods, and not of precipitate campaigns trying to convince the scientists to change their profession, converting themselves into the “innovation champions”. While the problem of healthy interaction between the fundamental and applied science is not yet solved, the situation was still more complicated by the recent progress of modern technology.

The informatics revolution

Some time ago, a persistent idea was that the youngsters cannot contribute to the public opinion. However, the informatics revolution abolished a lot of mythologies. Today, the teenagers have their personal lives and personal opinions. Together with young adults, communicated by blogs and twitters, they form a volatile mass with high capacity of mobilizing, either with constructive or destructive aims. At the moment, the e-revolution had some spectacular effects, such as the collapse of several authoritarian regimes. Yet, it also awoke fears... The world administrations are scared. Under various arguments (e.g., the copyright defense) they try to introduce the global inter-net censorship. The copyright problem of course should be solved, but without affecting the freedom of communication. If not, then by intensifying the press and internet controls, the state bureaucracies can create a dark, neo-totalitarian future exceeding even the fantasies of Aldous Huxley [7] or George Orwell [8], or hybrids of both [9]. In fact, if our recent crisis proves something, it shows that the truly weak point of the bureaucratic system is not an insufficient control of the human masses (including the scientists) but rather the complete lack of control on the upper social levels (banks, governments, parliaments, etc.). So, perhaps, the e-revolution could be the needed equilibrium factor?

The Anti-parliament?

Indeed, while our adult generation may be too busy or too tired, the blogging and twitting masses of youngsters, even without legislative powers, form already a world Forum, a kind of Anti-parliament able to identify the symptoms of our social diseases. Given enough time, some talented youngsters, instead of the dangerous sport of hacking might make their contribution to the future, collecting data about our structural and legal problems, detecting cases of bureaucratic and legislative nonsense all over the world. Indeed, it would be excellent to establish the Guinness records and prizes for the most talented absurd hunters!

A tempting idea would be also to create the archives of the *bureaucratic abuses.* In fact, in all countries the public life is infested by excessive demands facilitating the work of the bureaucratic apparatus, but the trouble caused by these demands exceeds massively the administrative gains they can bring. The detailed archive of such redundant laws would be of significant help.

In fact, one can only wonder, how could it happen that amongst enormous variety of sociological sciences, the studies of the *bureaucratic pathology* are still missing in the research institutes?

We are aware that many ideas presented here are not precisely defined, yet, they might be useful to defend some residues of our freedom. We live in a turbulent epoch of early prehistory, facing the challenges which only the future can resolve. Our Anti-bureaucratic web-page will be open for your opinions and ideas. Are we ready to say: *Vive la liberté?* Best regards.

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